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Estimates on degenerate jump-diffusion processes and regularity of the related valuation equation

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Abstract Many risk-neutral pricing problems proposed in the finance literature require to be dealt with by solving the corresponding Partial Integro-Differential Equation. Unfortunately, neither the standard Sobolev spaces theory, or the present literature on viscosity solution theory is able to deal with some problems of interest in finance. A recent result presented by Costantini, Papi and D'Ippoliti accepted for pubblication on Finance and Stochastics [17], shows that, under general conditions on the coefficients of the stochastic integro-differential equation, whenever a Lyapunov-type condition is satisfied, the stochastic process does not reach the boundary of the domain where is defined. Furthermore, in the same work it has been proved that there exists a unique viscosity solution to the pricing problem when we deal with the corresponding pricing problem for European-type derivatives. The viscosity solution theory ensures just the continuity of the solution, when data are continuous, but does not guarantees that such a solution has some additional regularity.

The aim of this work is to improve, for the pure differential case, the results existing in literature dealing with the regularity of both the solutions X of the underlying stochastic differential equations, and the solutions of the corresponding PDE. In particular we will provide some estimates related to dependence with respect to the initial data for the process X. Furthermore, dealing with the pricing problem, we improve our understanding on the assumptions that ensure the viscosity solution to have additional regularity properties beside the mere continuity.

A Lipschitz-type dependence result with respect to initial data, until a stopping time τ, is shown whenever the coefficients are locally Lipschitz continuous, and a Lyapunov-type condition is satisfied. Such a result can be improved if a suitable weight function is put in place.

A standard result in PDE theory ensures that, if the assumptions we assume in our work are satisfied, then in each compact subset where the diffusion matrix is positive defined, there exists a unique classical solution to the localized problem if initial data are continuous (see e.g. [35] or [9]). We make use of such a result in order to prove that this classical solution coincides, in the same subset, with the unique viscosity solution found in [17].

We give an application of such results, applying our evidences to the stochastic volatility model proposed by Ekströ̈m and Tysk in [29]. In such a case all the hypotheses we are currently assuming are satisfied, and the expression of the Lyapunov function can be explicitely provided for different final payoff. As a consequence, we are able to get the results of the existence and uniqueness of a classical solution to the pricing problem presented in

[29] in an independent way. Furthermore it is possible to consider weakened assumptions on the final payoff. On the other hand we try to consider a generalization of the model, allowing the process exhibits sudden jumps provided that the jump measure satisfies some suitable properties. In such a case, the expression for the Lyapunov function is provided as well, hence we are able to state that the considered valuation problem admits one and only one viscosity solution.

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Chapter 1

Introduction

It is well known that when we deal with the financial problems, several approaches may be used. In principle, in order to give a description of the evolution of the markets and the prices of the products that are traded in, a fundamental approach should represent an useful approach. This means that the price of the products could be derived by economical arguments, and then it could be obtained by matching demand and supply curves. On the other hand, all the results critically depends on our expectations on the behaviour of the players, and the set of available data. When the considered market is not liquid enough, or we are in a holygopolistic regime, such an approach seems to be the best way to perform some analysis on those markets, since the behaviour of a given player strongly influences the dynamics of all the market. Such influences may be very different from the ones observed in the past.
However, as far as now, all the financial markets around the world, and most of the commodity and energy markets are fully liberalized and very liquid, and the number of players is very high. In such conditions it is quite impossible to give a correct description of the behaviour of each player. Hence a parameterization of the dynamics becomes preferable, and the stochastic theory can be used with success.

Since the work made by Itô in [46] on the stochastic integration, several works have been developed reguarding martingales, semimartingales, and Lévy generators in general. The importance of the results obtained in these works is the reason for the success of the stochastic theory on the description of the market theory. In particular, many problem of interest in finance deal with the pricing of a given contract, once the model assumed for the evolution of the products traded in the market is chosen. In particular, the existence of the so called *fair price* of the contracts is of main interest, as well as the uniqueness of such prices. The fair price of the contract is commonly considered to be the expected value of such contracts, despite several difficulties may arise when some choises for the market have been done. Furthermore the uniqueness of the price may be not ensured.

Actually, two main ways are followed in pricing deriviatives. The most natural is to consider the fair price of a contract as the expected value at expiry, actualized by a discount factor. However, dealing with the stochastic differential equations directly may be very hard, since the distribution of the prices in the future is usually unknown. Even numerical procedures fail in some cases. On the other hand, it is well known that the problem of pricing a contract may be addressed by dealing with the corresponding Partial Integro-Differential Equation (PIDE) related to the considered market and the particular type of contract.

Unfortunately, in many cases of interest, the considered problem has degenerate diffusion matrix, or the coefficients are fast growing at infinity, or may exhibit other features under which the present literature is not developed enough in order to guarantee useful results reguarding the existence, uniqueness and regularity of the solution to the PIDE.

This thesis is based on a joint research with Marco Papi, and starts from a forthcoming work presented by Cristina Costantini, Marco Papi and Fernanda D'Ippoliti [17] that generalizes the present results on the existence and uniqueness of the viscosity solution for many problems of interest in finance. Roughly speaking, such hypoteses require that the coefficients of the stochastic differential equation with jumps are locally Lipschitz continuous, and a Lyapunov-type condition is verified. Then, the aim of our work is to improve such results, and some regularity of the stochastic processes is found. In particular, under the same hypoteses assumed in [17], that ensure the existence and uniqueness of the viscosity solution, the stochastic process is found to be Lipschitz continuous with respect to initial data, when a stopped problem is considered, or if the drift term is globally Lipschitz continuous. Furthermore, starting from such results of regularity of the stochastic process, it is shown that strong implications on the regularity of the viscosity solution are available. In particular, in each compact set K where the final payoff is continuous and the diffusion matrix is locally positive defined, then the viscosity solution is not only continuous but inherits some additional regularity. In particular we are able to prove that the viscosity solution is twice differentiable with resepct to x and once with respect to t for each $x \in K$. Such results are applied to our generalization of the model proposed by Ekström and Tysk in [29].

This work is organized as follows. Chapter 2 is devoted to give an introduction of the market theory and the basic financial concepts. Following mainly [9], [3] and [51], we provide the standard mathematical definitions of the market, the players acting in the market and the main products that are traded. Then a description of the models mainly used in financial world is given, highlighting the main features and drawbacks. At the end, the aim is to introduce the arbitrage principle highlighting which are the implications in financial mathematics.

Chapter 3 is devoted to indicate the *state of the art* in pricing problems and the main problems of interest in finance. Our attention is mainly pointed towards such problems to which the present literature cannot be applied.

Chapter 4 begins with a very short introduction to the semigroup theory expecially applied to the martingale problem, highlighting the links between the semigroup approach and the Markov processes. Then the main assumptions that hold for the rest of the work are shown, and the results of well-posedness of the martingale problem $(\mathcal{A}, P_0)$ for any initial distribution P_0 provided in [17] is presented. Such a result is obtained under very general hypoteses on the operator $\mathcal{A}$. Then, in the last part of the chapter, under the same assumptions given in [17], some estimates of continuity with respect to the initial data for the process X are provided in the pure differential case. In particular we are able to give a Lipschitz-type dependence of the process X with respect to x and t in a suitable sense. Furthermore, a stronger result is provided as well whenever a suitable weight function is considered.

Chapter 5 introduces the problem of existence and uniqueness of the solution for a very simple case of singular valuation equation. Then, the general pricing problem is considered. The main results provided in [17] are then proposed. In particular it is shown that under suitable assumptions there exists a unique viscosity solution the general problem. Then applying the results in Chapter 4, it is possible to show that the viscosity solution u is not only a mere continuous function but, in each compact subset where the final payoff ϕ is continuous and the running cost f and the interest rate c are α-Hölder continuous, and the diffusion matrix is uniformly positive defined, the viscosity solution is twice differentiable with respect to x, and once with respect to t.

Chapter 6 is devoted to give an application of the results of regularity for the pure differential case got in the previous chapters. In particular a focus on the model proposed by Ekström and Tysk in [29] is given, and the main improvements with respect to the previous models is presented. Then, we show that, under an additional assumption on the behaviour of the coefficients near the boundary, the model proposed by Ekström and Tysk satisfies all the assumptions previously hold, and admits a Lyapunov-type condition. Hence, all the results got in the previous chapters can be applied. In particular the existence and uniqueness of a classical solution is ensured for a larger class of final payoffs than the ones considered in [29]. Furthermore, a generalization of the model is proposed, assuming the stochastic process exhibits jump-diffusive dynamics. In this case, it is possible to find a viscosity solution to the integro-differential problem when the final payoff does not blow up fastly at infinity. Furthermore, in the case the pure diffusive problem is considered, then the existence of a unique classical solution is ensured.

1.1 Comments and notation

The following notation and convention will be used in the thesis, where not differently specified.
The symbol (x, t) denotes a point in the space $D \times [0, T]$ where $D \subseteq \mathbb{R}^d$ and $T \in \mathbb{R}_+$, unless differently specified, and x_i represents the i^{th} element of the vector x.
The partial derivatives with respect to t and x_i will be denoted by ∂_t, and ∂_i, or equivalently ∂_{x_i}. When the derivative is applied to a function $u(x, t)$ then the derivatives could be equivalently indicated by u_t or u_{x_i}.
The symbols ∇ and ∇^2 denote respectively the gradient and the Hessian matrix, and are applied to spacial coordinates. The gradient of a function ∇u is considered as a row vector in $\mathbb{R}^d$.

Given a matrix A, the symbols $tr(A)$ and $\det(A)$ denote respectively the trace and the determinant of the matrix, while the superscript $\top$ denotes the transpose.

For any set $S \subseteq \mathbb{R}^d$ we indicate by $\overline{S}$ its closure, and by S^c its complement in $\mathbb{R}^d$. Instead the function $\mathbf{1}_S$ denotes the indicator function, that is $\mathbf{1}_S(x) = 1$ if $x \in S$, and $\mathbf{1}_S(x) = 0$ if $x \notin S$.

For any open set $S \subseteq \mathbb{R}^d$, $\mathcal{C}^{2,1}(S \times [0, T])$ denotes the set of real valued functions that are twice differentiable with respect to x and once with respect to t and the derivatives are continuous, while the symbol $\mathcal{C}(S \times [0, T])$ denotes, as usual, the set of real valued functions continuous with respect to their arguments. The symbol $\mathcal{C}^\infty(S \times [0, T])$ denotes the set of smooth real valued functions.

Given a metric space S, the calligraphic uppercase symbols $(\mathcal{A}, \mathcal{B}, \dots)$ denote a linear operator, if not differently specified.

The symbol $|| \cdot ||_p$ denotes the standard p norm that is, if h is defined on a Banach space S, then

$$||h(x)||_p = \left(\int_S |h(x)|^p dx \right)^{1/p}.$$

When the usual norm in the Euclidean space, where $p = 2$, then the index p is omitted.

For $D \in \mathbb{R}^d$ we denote by $L^p(D)$ the Banach space of measurable functions h defined on D such that $||h||_p < \infty$, and $W^{n,p}$ the Sobolev space of functions whose the first n derivatives belong to $L^p(D)$.

For any Borel set $S \subseteq \mathbb{R}^d$, $\mathcal{B}(S)$ is the Borel σ-algebra of S, and $\mathcal{M}(S)$ denotes the space of Borel measures on S, endowed with the weak convergence topology.

For any metric space S, $\mathcal{D}_S[0,\infty)$ denotes the space of right continuous processes that admits limit at left (cádlág), endowed with the Skorohod topology.

The symbol X denotes the solution to a given stochastic integro-differential equation, while the symbols X^x and X_t^x denote respectively the solution to a stochastic integro-differential equaiton starting from x and the solution to a stochastic integro-differential equaiton starting from x observed at time t. The symbol $\{\mathcal{F}^X\}$ denotes the filtration generated by the stochastic process X. We use the notation $\mathcal{L}^p$ to indicate the space of the processes X that are adapted to $\mathcal{F}_t$ whose paths are in $L^p(0,T)$ $\mathbb{P}-$almost surely.

Chapter 2

Preliminaries

2.1 Financial market in a mathematical framework

In last decades several works have been made in order to describe, in a rigorous mathematical way, the behaviour of the markets, mainly financial and commodity ones. In particular two main approaches have been developed, and both of them give a stochastic description of the markets as a suitable probability space. This is the space of the possible scenarios, endowed with a filtration that defines the set of the informations available at each time t. Despite this line on contact, these two approaches are opposite one to each other. The first approach is represented by *market theory* that, roughly speaking, explains the movements of the markets describing the evolution of the products that are traded in that are assumed to evolve following a given system of stochastic differential equations, possibly relating one or more assets. The value of such products determine the market itself. In such a theory, the actors are represented by the agents that are identified by their position in the market and their amount of share of a given asset. The goal of such an approach is to find the stochastic process that gives the best fit to the evolution of the market, and then to get as many informations as possible about the future in terms of probability distributions.
The second approach is the so called *economic theory*. According to such a theory, the attention is paid to the agents themselves and not just to their position in the market. In particular, each agent is characterized by given preferences towards consumption and their own total wealth. All these features are collected in the so called *utility function* that is specific for each agent, or group of them. Then the evolution of the prices of the products traded in the market and the position of each agent in the market are determined by the interactions between each agent, given by their utility functions. Clearly, such an approach better performs when the description of the preferences of the players, and their interactions can be easily explained by a mathematical approach.

In order to give a description of the evolution of the financial and commodity markets, it is clear that following an approach that takes into consideration fundamental arguments should give, theorically, the right behaviour of the considered markets. Then the price of the products traded in these markets could be derived by economical arguments, and the final price could be then obtained by matching demand and supply curves. Such an approach allows to perfome several analysis in details. However, it requires a very strong knowledge of the considered market, both from an economical point of view, and, sometimes, even from a technical point of view. For such reasons a parametrization of the process is often preferable, and the stochastic process theory can be used successfully, expecially when we are dealing with the problem of pricing derivatives. In particular, more the markets are liberalized and liquid, more a stochastic approach can be succesfully applied. Such a parametrization can be obtaind directly looking at the behaviour of the market. Hence, it is easy to argue that the market theory approach is suitable when the number of agents is large, instead the economic theory approach is particulary suitable in an holygopolystic markets, such as some electricity markets. In the first case, indeed, given the complexity of the problem, it is quite impossible to find the utility functions for each agent and to understand the influences and interactions between the agents that may take place. This fact is true expecially since the preferences of the players may change during the time. A stochastic approach, instead, represents a kind of parametrization of the behaviour of each agent.

On the other hand, when the main players in a market are few, it is possible to overcome this challenge and the economic theory approach seems to be more feasible. Furthermore, in this case the behaviour of the market should be strongly determined by the choices of the main agents, then it is important to take into account properly the preferences of such a player. This fact can be easily achived following the economic theory, that is expecially indicated in large investor theory in the case of financial markets, and in holygopolystic markets in the case of commodity ones.

In the next sections we recall some standard results of the market theory, in order to introduce the approach that will be usefull for our work. Furthermore some of the mostly used models in the financial world are presented. The chapter is organized as follows. In the first two sections we provide the standard definitions of the market in a mathematical framework, the players acting in the market and the main products that are traded in. In Section 2.4 a description of the models mainly used in financial world are provided, highlighting the main features and drawbacks. In the last section the arbitrage principle is introduced and some standard results on the completeness of the markets and the pricing by hedging are provided.

2.2 An outlook on market theory

Now we rapidly introduce some fundamental aspects and definitions that are put in place in market theory.
As argued before, the randomness of the markets can be substantially addressed to the fact that the behaviour of the players acting in the market is not predictable. Then modeling financial markets making use of the stochastic process theory seems to be the most natural way. This fact is particularly true if the behaviour of one of them does not strongly affect the choises of the other players and the dynamics of the prices.
Due to its randomness, from a mathematical finance point of view, a market can be described as a probability space, that represents the set of the possible "worlds", the informations that are available at any time, and the measure of the probability that weights the occurences for each "scenario". Furthermore, the mathematical definition of the products that can be traded in the market is also needed.

Definition 2.1. *For any $T > 0$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed by a filtration $\{\mathcal{F}_t : t \in [0, T]\}$, such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$, is a* ***market place*** *of duration T.*

Such a definition allows us to use a probabilistic approach in order to describe the market. Indeed in Definition 2.1 the element Ω is the "space of the trajectories" and represents the set of all events that can occur. The term $\mathcal{F}$ is a σ-algebra of the parts of Ω, and $(\mathcal{F}_t)_{t\geq 0}$ is the filtration, that is a family of sub-σ-algebras of $\mathcal{F}$ and is increasing with respect to t. From an intuitive point of view it represents the set of the events of which it is known their occurence until the time t. It can be seen as the set of the informations that are available at time t. At the end the term $\mathbb{P}$ is the probability measure of each part of Ω, then the probability that a given set of events occurs.

In order to introduce the definition of the market, we define, from a mathematical point of view, the main products available in each specific market. In particular we refer to stocks, commodities, indexes, bonds, and so on. As we see, such products are assumed to evolve, in general, following some known stochastic processes. It is possible to distinguish two classes of products. The first one is represented by the riskless products

Definition 2.2. *A* ***bond*** *is a riskless asset. The price $S^0 \in \mathcal{L}^\infty$ of the bond is assumed to evolve following the equation*

$$dS_t^0 = c_t S_t^0 dt, \tag{2.1}$$

where c_s is the instantaneous ***interest rate****.*

We observe that c_t is in general a stochastic process. The value of the bond can be seen as the price of a bond that will be surely refounded at exipiry. We remark that if the instantaneous interest rate c_t in the equation (2.1) is a deterministic function of the time $c(t)$, then the process S_t^0 is deterministic too.

At the other hand, we can define a risky asset as a stochastic process S_t

Definition 2.3. *A **stock** is a risky asset whose value $S_t \in \mathcal{L}^2(0,T)$ is assumed to evolve following a given stochastic differential equation.*

In such an environment, the word stock is used in order to indicate the value of any product that can be traded in a market. In particular, for general purposes the following quantity is defined

Definition 2.4. *A stochastic process $X(\omega) \in \mathbb{R}_+^d$ such that for some $p \geq 1$, $X(\omega) \in \mathcal{L}^p(0,T)$ is the **price of an asset** if it is adapted to the filtration $\mathcal{F}$.*

Clearly, such a definition includes both the case of riskless assets and the risky ones.

The bond defined in Definition 2.2 is usually used in order to actualize the price of the returns X. Often it is referred to as *the price of time*. The actualized price $\tilde{X}_t^k = X_t^k/S_t^0$ is the future value of the asset X^k discounted for a riskless interest rate. In other words this is the wealth that one agent has to invest at time $t = 0$ in a riskless asset, in order to receive X_t^k at the instant t. Such a definition explains the expression *price of time* used before.

Then the definition of the market is given as

Definition 2.5. *A couple composed by a market place and a vector $X = (X^0, X^1, \dots, X^n)$ of assets is a **market** $\mathcal{M}(X)$.*

Usually, the first term X_t^0, stands for the money market account S^0. The price of a stock is allowed to be a process, evolving following any stochastic differential equations, provided that Definition 2.3 is satisfied. In particular, different choices of the stochastic differential equation followed by the process X is the difference between different market models. Furthermore, once the model is chosen, the process X_t evolve following known laws in the stochastic sense. Since different choises for the market model give different behaviour of the assets' prices, it is important to remark that such models have to be compliant with some assumptions that are commonly accepted in the financial world. In particular, it is common to accept the assumptions that all the the informations about the assets are reflected directly in their actual values. This fact reflects the markovian property of the market.

2.3 Financial derivatives

As pointed in Definition 2.5 what differentiates a market model from another one, is the choice of the stochastic process assumed for the evolution of the assets $X(\omega)$. In particular they define the market model we want to consider, and determine all the features of the market. Before presenting the stochastic processes most used in financial and commodity markets, we deal with some products that can be traded in traditional markets. Indeed, in addition bonds and stocks defined in Definitions 2.2, 2.3 and in general in Definition 2.4, there is an important class of products available in the market. Their final value is not determined directly by the transactions of such products rather than the price of other underlying assets X_t evolving in the market. For such a reason, such kind of products are called *derivatives*. When we speak about derivatives we usually refer to as assets. From a mathematical point of view, such a definition is not properly used in the sense of Definition 2.4. Indeed, as we see, their value is completely determined knowing only their final value at time T, but it is not needed to specify any stochastic process followed by the underlying product. For this reason, it is not clear if they can be seen as a process, even if such contracts a written on some risky underlying.

These products was born in financial and commodity markets in order to offer a kind of hedge to the utilities against sudden movements of the market. As an istance we can imagine that an airline company may purchase, at a fixed price, a supply of jet-fuel for the future. In such a way the company can freeze the future expenditure avoiding sudden movements in the markets cause additional losses for the company. The same statement is valid for the supplier as well. However, during the years such products became more complex and they started being used with speculation purposes. Some of such kind of products and their properties are presented in [43]. Some of them are strictly specific for some market, such as currency or interest ones, but, generally speaking, the derivatives are quite common in all the markets.

From a mathematical point of view, a derivative can be defined as

Definition 2.6. *Given $T > 0$ and a vector of assets $X_t(\omega)$ defined on a domain D, a **contigent claim** or **derivative** on X is a couple (T, ϕ), where $\phi \in \mathcal{C}(\overline{D}) \cap W^{1,\infty}_{pol}(D)$, $\phi \geq 0$ is the **payoff**.*

It is possible to identify mainly three kind of derivatives. The most simple is the **European** type derivative, that may only be exercised at expiry. In order to guarantee more flexibility to the owner of the right, has been subsequentially proposed a kind of derivative that can be exercised at any time before the expiry. Such kind of derivative is defined **American**. A third family of derivative is represented by **Asian** one. This kind of derivative is very common expecially in the commodity markets, where the price of the contract is determined as an average, on one or more underlying products, of the prices along a fixed interval of time.

We have already introduced one of the question concerning derivatives, that is how is possible to find the fair value of one contract, given all the features assumed for the market, and expecially if there exists such a value and if it is unique in the market. From an economical point of view the answer obviously depends on the expectations that the traders have about the market movements, and their adversion to the risk. On the other hand, we have to consider that in order to enter a derivative contract a "fee" have to be paid. Such a value can be seen as the price of the contract. Such a statement is clear if we think about an option contract that gives to the owner just rights without any obligation. From a mathematical perspective, the existence and uniqueness of such a price depends on some technical aspects that can be ascribed to the considered market model and the particular derivative. Hence, it is not clear, in general, if there exists for any time t a unique price for the derivatives. On the other hand, despite the market model can be chosen in order to give positive answers to this question, however the stochastic process describing the market should be realistic enough to reproduce as best as possible the behaviour of the prices of the underlying assets.

In order better to understand the concept of derivatives, let us make some examples, beginning from the most simple one that can be represented by the **forward contract**.
Under a forward contract one agent agrees to sell to another agent some commodities, financial assets, or assets in general, at a specified *future date* for a specified *delivery price*. Such kind of contracts can be, for example, signed by an airline company and a jet-fuel supplier, in the case the airline company wants to hedge herself by sudden rises on jet-fuel prices. We remark that in such a case, the derivative is used to hedge the company and not with purposes of speculation. We remark that one of the most important thing that the two parties have to fix is the *fair* future price. In a mathematical framework, if X_t^1 is the only product underlying the contract, T is the future date and F is the delivery price, then the considered derivative is given by:

$$\phi^{fwd}(X_T^1) = F_{T,t}$$

with T the time to expiry.

More recently, some derivatives have become very popular since their flexibility, and the progresses in mathematical finance. As an istance, such kind of instruments is given by **options** on stocks, commodities, currencies, and so on and so forth. The most simple example of such kind of contracts are very similar to forward ones. The main difference of the options with respect to the forward is that the owner of these securities has the right, but not the obligation to exercise the contract. The most famous options that can be traded in the regulated and not-regulated market are **call** and **put** options. A call option on a stock S is a contract which gives to the owner the right to buy at a fixed *strike price* K one unit of S at a fixed future date T. Obviously the owner of the contract will exercise the right just if the strike price K is lower than the value of the stock in the market at time T. In such a case it is easy to derive the payoff as

$$\phi^{call}(S_T) = (S_T - K)_+.$$

A put option, instead, gives to the owner the right but not the obligation to sell a stock at a given price K, in a fixed date in the future. In such a case instead, the payoff is given by

$$\phi^{put}(S_T) = (K - S_T)_+.$$

Consider for example the airline company entering a contract of a future supply of jet fuel. She may decide to sign a forward contract or a call option, with the same strike price K. It is clear that a very important issue is to find the fair price of these products, so that, if the market is "efficient", the company should be indifferent to enter the forward contract or the call option. If it were not the case, it should be possible to get some profit without any risks, and all traders in the market would follow a strategy such that the prices will move in order to eliminate such opportunities of earning.

More exotic derivatives are also traded in regulated and Out of The Counter (OTC) markets, such as **options on options**, **barrier options**, **asian option**, **look-back option**. All of these kind of derivatives was born in order to hedge the owner of the rights against sudden movements of the markets. For example asian options are used in order to mitigate the high volatility of some underlyings. In these contracts, for example, the value of the payoff may be not fixed at the beginning, but it can be determined as an average of the underlying. In this case, if S_t is the price of a given underlying, we can define the process Z_t as

$$Z_t = \int_0^t S_r dr.$$

Then the value of the payoff may be given by

$$\phi^{as}(S, Z) = (S_T - Z_T)_+.$$

Some other kind of asian option can be defined as well. See for example [65].

Another class of problems that takes place when we deal with derivatives comes from the fact that the underlying products evolve during the time. As a consequence, more the time flows, more the value of the derivative, if exists, can change with respect to the initial istant. Then we may be interested in understanding if there exists a "strategy" such that, purchasing and selling the products of the market we are able to fix the value of the contract, or to minimize the risk associated to the contract. Reguarding this specific problem, additional difficoulties come when we deal with large investor economies, or holygopolistic markets. In such cases, indeed, the hedging strategy of the investors may affect directly the price of the derivatives, and then even the value of the risk-free interest rate.

In particular, the strategies followed during the time allow to identify the traders considering their position in the market, given their portfolio, whose value is given by its total wealth.

Definition 2.7. *Given a market $\mathcal{M}(X_t)$, a process Δ_t with values in $\mathbb{R}^{d+1}$ is defined **dynamic strategy** if $\Delta_t^k \in \mathcal{L}^{q_k}(0,T)$ and it is $\mathcal{F}_t$-predictable.*

The k^{th} element of the process Δ_t represents the number of shares of the k^{th} asset hold by the trader at time t. In particular the element Δ_t^0 represents the share of the riskless asset hold by the trader.

Definition 2.8. *A strategy is **self-financing** if its total wealth $\Delta_t \cdot X_t$ satisfies the condition*

$$\Delta_t \cdot X_t = \Delta_0 \cdot X_0 + \int_0^t \Delta_s \cdot dX_s \tag{2.2}$$

*where the integral operator has to be intended in the Itô's sense. If such a strategy has a non negative total wealth for each $t > 0$, that is $\Delta_t \cdot X_t \geq 0$, it is called **admissible**.*

Such a definition formalizes the fact that the trader does not add or subtract any value to his portfolio, but the evolution of its total wealth is determined only by the movements of the assets that belong to the portfolio.

What carachterizes the behaviour of the traders is their expectation on the movements of the market. From a mathematical perspective it is reflected in the specific stochastic process assumed for the evolution of the prices.

We have already said that one of the problems of interest in finance is to understand if exists a strategy that minimizes the risk associated to a contigent claim. From an economical point of view, an **hedging strategy** is represented by a dynamical selling and purchasing of the underlying, so that the value of the contract is constant untill the maturity, and the risk associated to the loss of value of the contract is minimized. If such a strategy there exists it represents a very useful mathematical tool even for pricing problems. Indeed, clearly, following the hedging strategy, the total wealth of the portfolio automatically gives the fair value of a given contigent claim at any time t.

Definition 2.9. *Given an European contigent claim (T, ϕ), an admissible strategy Δ_t is an **hedging strategy** if*

$$\mathbb{P}(\Delta_T \cdot X_T = \phi(X_T)) = 1.$$

For any time $t \in [0, T]$ the quantity defined as

$$Y_t^* = \inf \left\{ Y_t = \Delta_t \cdot X_t : \Delta_t \textit{ is a hedging strategy for } (T, \phi) \right\},$$

*is named **value by arbitrage**.*

We remark that, by the definition of the total wealth of a trader, the value Y_t^* is exactly the total wealth of the portfolio of a trader.

On the other hand, several works have been proposed, dealing with both the pricing problem and the hedging problem. In particular important results have been obtained reguarding the existence and uniqueness of a pricing function $u(X_t, t)$. However it is important to remark that the existence and uniqueness of the solution to the pricing problem strongly depends on the particular considered market model, and the specific hypotheses that are made on the evolution of the assets.

2.4 General features of the markets and stochastic processes

During the years many stochastic processes have been proposed in order to describe the behaviour of financial products and commodities, such as interest rates, exchange rates, crude oil price, gold price, and so on and so forth. All these products are very different one to each other. Despite that, looking at the returns of the products traded in such markets it is possible to see that some features are common in all of them, at least as a first approximation. Then, the stochastic models that have been developed are very similar among the different markets.

As we have already mentioned, which differentiates a market theory from another one, is the stochastic process assumed for the evolution of the products. However, all the models have to respect some financial principles. Such principles are reflected directly on particular assumptions, that have some implications on the models that can be chosen.

On the other hand, the stochastic model have to fit several features in order to better describe the behaviour of the market. In particular it is important that the model used is able to describe as best as possible the features of the observables in the market, among the most important being fat tails, seasonality, sudden jumps in the prices, skew in volatility and so on and so forth. Obviously, more the model used is complex, more it is able to match the behaviour of the real market data. The drawback is that the analytical tractability is lost when such a complexity increases, and standard results on existence and uniqueness of the solutions for the considered problems may be lost.

Affine models. The mathematical tractability of one model is reflected directly when we try to evalue financial derivatives. In such a case it is obvously preferable to have closed-form - or a semianalytical expression for such prices. This is one of the reasons for the success of affine models. Indeed, in an affine model, if the "discount rate function" is affine and the final payoff is of affine or exponential-affine form, the solution of the risk neutral valuation equation for a European type derivative can be found by solving a system of ordinary differential equations in the Fourier space K, and then inverting Fourier transform. Furthermore, in some cases, the solution of such differential equations is explicitely known.

The class of Affine Term Structure Models, introduced by Duffie and Kan in [24], combines some financial appealing properties

1. the sensitivities of the zero coupon yield curve to the stochastic factors are deterministic
2. explicit parametric restrictions imply the existence of an affine process (see e.g. [20])
3. the pricing problem can be reduced to the solution of a system of ordinary differential equations as discussed in [25].

The core of the affine term structure models is the framework of Duffie and Kan [24]. These models have been employed in finance since decades, and they have found growing interest due to their computational tractability. There is a vast literature on affine models of which we mention just few research works [24], [20], [25].

In the case of Affine Models the process is assumed to evolve according to a pure diffusive stochastic differential equation, where the coefficients $\mu(x)$ and $\sigma(x)$, that are respectively the drift and the diffusive matrix of the process X_t, can be written as

$$\sigma(x)\sigma^{\top}(x) = A + \sum_{i=1}^{d} x_i B_i, \qquad \mu(x) = \alpha + \sum_{i=1}^{d} x_i \beta_i. \tag{2.3}$$

The terms A, B_i are $d \times d$ real matrices and $\alpha,\ \beta_i \in \mathbb{R}^d$, for $i = 1, \dots, d$. As discussed in [20], model parameters cannot be chosen arbitrarily, but there are admissibility restrictions required for the existence of the process X_s. The authors prove the existence, for each value of d, of $d+1$ disjoint admissible regions of the parameter space.

In each of these families, different restrictions are imposed on the parameters. Moreover, affine models do not have a unique representation, that is, there exist different choices of the model parameters that generate identical behavior of the process.

Although the affine models have a simple mathematical tractability, often they are not complex enough to describe the behaviour of the products traded in the markets. Furthermore, even if the model used to describe the evolution of the market is affine, it is possible to find several problems, including many used in financial world, that cannot be formulated with affine or exponential affine final payoff. An example of that is given by Asian option in some stochastic volatility models.

Black-Scholes model On the other hand we may suppose that the stock prices evolve following the *Black and Scholes* model proposed for the first time in [15]. Such a model had a huge fortune in the financial world, since it is handy from a mathematical point of view, and allows to give several explicit results for almost all problems in terms of probability distributions and calculation of the moments, expecially when pricing of derivatives is considered. Furthermore, such a model has been derived following some economical arguments (see for instance [43]). In particular the value of the bond is driven by a known constant interest rate r, transaction costs associated with hedging a portfolio are negleactable, it is possible to perfome short selling, and the assets are infinitely divisible.
Each of these assumptions has important implications on the considered market. As an instance it is possible to consider a stochastic process with continuous paths of the stock price $t \mapsto S_t(\omega)$ for almost all $\omega \in \Omega$ at least a $\mathbb{P}$-negleactable set. The absence of the transaction costs, and the infinitely divisibility of the assets mean that a hedging strategy in a sense that will be given later may be followed purchasing and selling continuously the asset without any additional costs for the trader.
In particular, the price of the asset is assumed to evolve following the stochastic process

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_t \tag{2.4}$$

where the coefficients μ and σ are constant, and W_t is a standard Brownian motion.

Despite that, such a model encounters relevant disagreements with the real world in terms of the distribution of the hystorical returns and forecasts. Expecially, Mandelbrot observed in [54] that the logarithmic of the returns of the price exhibit a long-tailed distribution. Such a feature cannot be explained by any exponential Brownian motion as Black and Scholes model. Furthermore it is not able to describe the mean-reverting trend exhibited for istance by commodities, FX, interest rates or sudden jumps that are present, for instance, in the power markets.
On the other hand, some problems arise in the results themselves. In fact, we remark that the volatility σ is supposed constant in such a model. When we evaluate a derivative such as a call option, the result obviously depends on the value of the volatility, the trend of the assets, and other parameters that depend on the particular contract. If we try to invert such a result, in order to express the volatility in terms of the other parameters, then we get a nonconstant expression for the volatility anymore, but it depends on the parameters we have

mentioned, and exhibits a sort of *smile*. The volatility that we get once we have inverted the value of the contracts, is the so called *implied volatility*. This paradox is known as *smile of the implied volatility*.

In order to overcome such issues several works have been proposed, and the theory in the field of stochastic processes have been developed. All the proposed models try to overcome such disagreements considering more complex dynamics. As an instance, multiple sources of uncertaninty such as sudden jumps, are considered in the evolution of the assets besides to the Brownian motion W_t.

Log-normal models. With the expression *log-normal models* we mean a class of models in which the process X is log-normally distributed. In such a kind of model, is included the one proposed by Black and Scholes. We have to remark that this class of model is of interest in commodity markets, where a mean-reverting effect is exhibited. In such a model, the prices are assumed to evolve following

$$dX_t = \kappa\big(\theta - \delta \ln X_t\big) X_t dt + \sigma X_t dW_t, \tag{2.5}$$

where δ may assume only the values 0 or 1. We observe that if $\delta = 0$ then it reduces to the Black and Scholes model. Taking into account the logarithm of the prices $Y_t = \ln X_t$, and applying Itô's lemma, then it is possible to show that the process Y_s is distributed following the gaussian law, and for $\delta = 1$ we get

$$Y_s \sim \mathcal{N}\left[e^{-\kappa(s-t)}Y_t + \Big(\theta - \frac{1}{2\kappa}\sigma^2\Big)\Big(1 - e^{-\kappa(s-t)}\Big), \sigma^2 \frac{1 - e^{-2\kappa(s-t)}}{2\kappa}\right], \tag{2.6}$$

with $s > t$, where $\mathcal{N}(\mu, \Sigma)$ indicates a normal distribution with mean μ and variance Σ. This fact allows us to use all the results valid for the well-known Black-Scholes model. We notice that such a model is able to describe several important behaviour observed in the real energy markets. Despite that, several features cannot be described by the this models such as fat tails, jumps and, in the case of the power, spikes. For such a reason, since the work of Black and Scholes, more complex models have been proposed in order to describe the behaviour of the real markets.

We have to remark that some of the features discussed above can be included increasing the complexity of the model in (2.5). This is the case when the parameters θ, κ and σ are non-constant but depend on the time, or are even governed by stochastic processes themselves. However, in such cases the model usually becomes too hard to be calibrated and very instable.

Local volatility and stochastic volatility models. It is well-known in financial literature that, if we assume the volatility be constant then the considered problem is simplified from a mathematical point of view of course. Unfortunately the behaviour of the market prices cannot be described well by such models. On the other hand, considering the volatility of the prices as a stochastic process as well allows us to gain more flexibility in describing trajectories of the prices, but several additional issues have to be considered. We see in the Chapter 6 that if the growth rate of the state space vector is not sublinear, then the uniqueness of the solution to the stochastic problem (2.9), may not be hold.

Several reduced-form volatility models have been presented in last decades, such as in [34, 40], that drops the Black-Scholes assumption of constant volatility. The main idea is to assume that the prices evolve following a process such as Black-Scholes, or more complicated model, such as mean-reverting ones, but the volatility σ is not constant anymore. The simplest assumption is to consider a volatility term $\sigma : [0 : \infty) \to [0, \infty)$ as a general function of X_t, usually being

$$\sigma(X_t) = \sigma_0 X_t^{\alpha}.$$

We notice that in the case α is grather than zero, then we have a positive correlation between volatility and the prices, negative otherwise. In this way we could take into account also the "inverse leverage effect" exhibited by the energy prices. Due to this effect, when prices rise up the volatility rises up too, instead, when the prices are low, then the volatility decreases. Such models are called "Constant Elasticity of Volatity Models", and they are well-studied for some values of α. These models may induce skews but not smiles in the volatility surface, then such simplified models are not well-suited for describing natural gas and crude oil prices, since they exibit both behaviours as showed in [30].

A particular form of the local volatility models has been given by Hobson and Rogers in their work in [41]. In such a model the volatility of the asset is not a stochastic process as the prices are, but is a function of the difference between the current price and the average of the prices seen in the past. From a mathematical point of view, under the risk neutral probability measure, the stock price is described by the following dynamics

$$\begin{aligned} dX_t &= rX_t dt + \sigma(\omega_t) X_t dW_t \qquad (2.7)\\ \omega_t &= \log X_t - \int_{-\infty}^{t} \frac{1}{\tau} e^{-(t-s)/\tau} \log(e^{r(t-s)} X_s) ds \end{aligned}$$

where the parameter $\tau > 0$. It is clear that the contribution of X_{t-s} to the valuation of ω_t can be neglected when $(t - s) >> \tau$. This means that the weight of the values far away in the past, is lower than the actual ones. From an economical point of view this fact is easily explained. Such dynamics may reflect some structural variations in the market that may cause a breakdown of calculation of the mean with respect to the past. A very clear example comes directly from the recent economical global crisis when, after 2009 all the historcal data observed until 2008 were not able to explain the behaviour of the marktes for future maturity.

Clearly, in general, the Hobson-Rogers model is not an affine model. Furthermore, if $\sigma^2(\omega_t)$ is an affine function of ω_t, then the dynamics of the Hobson-Rogers model becomes a special case of the Heston model.

It is possible to improve the Constant Elasticity of Volatility models allowing the volatility of the prices to be itself a stochastic process V_t. The most general form that can be allowed for a stochastic volatility model is given when the prices and the volatilities are assumed evolve following such relations

$$\begin{aligned} dX_t &= \mu(X_t, V_t)dt + \pi(X_t, V_t)dW_t^1 \\ dV_t &= \beta(X_t, V_t)dt + \sigma(X_t, V_t)dW_t^2 \end{aligned} \tag{2.8}$$

where W^1 and W^2 are two possibly correlated Brownian motions, with correlation ρ. However, we have to remark that the sotchastic volatility model such as (2.8) may be very hard to be analytical managed. Furthermore, the existence and uniqueness of the solution (2.8), or the pricing problem of a derivative written on an underlying evolving as (2.8) may not be hold. Then a simplification have to be done.

Perhaps, the most popular stochastic volatility model in financial world is represented by Heston model. Its popularity is due to the fact that it is quite handy and it is possible to have a closed-form solution for some simple derivatives. Such a model can be obtained starting from (2.8) where the prices are assumed to evolve as a Black-Scholes model and the volatility is assumed to evolve following a Cox-Ingersoll-Ross (CIR) model (proposed the first time in [18]), then

$$\begin{aligned} \mu(X_t, V_t) &= \mu X_t, & \pi(X_t, V_t) &= \sqrt{V_t} X_t \\ \beta(X_t, V_t) &= \kappa(\theta - V_t) & \sigma(X_t, V_t) &= \sigma_0 \sqrt{V_t}. \end{aligned}$$

Correlated interest rate factors. In the last years, Bernaschi et al. proposed in [14] a n-factor term structure model for valuing public debt securities where each factor follows a CIR type model, and the driving Brownian motions are correlated.
It is well known that an unidimensional CIR process is affine, however, when a correlation matrix between the Brownian motions is considered as well, the d-dimensional factor process is not affine anymore. Indeed the diffusion matrix have mixed terms that, in general, contain the square root of the product of two interest rate processes. As an instance, in the

case of a bidimensional process the diffusion matrix $a(r_t^1, r_t^2)$ takes the form

$$a(r_t^1, r_t^2) = \begin{pmatrix} \sigma_1^2 r_t^1 & \rho\sigma_1\sigma_2\sqrt{r_t^1 r_t^2} \\ \rho\sigma_1\sigma_2\sqrt{r_t^1 r_t^2} & \sigma_2^2 r_t^2 \end{pmatrix}.$$

It is clear that the model considered by Bernaschi *et al.* does not fit the class of affine model of r_t^1 and r_t^2 if not for the special case where $\rho = 0$.

Jump-diffusion models These kind of models were firstly proposed by Mandelbrot [54]. After that, several models have been proposed, starting from the ones proposed by Samuelson in [60] and [61]. These models belong to a general class of stochastic processes that are called Lévy models. Such models have been recently studied and an exstensive literature on these processes is available, of which we mention just few works such as [26, 27, 36, 55]. However, for our purposes, we will focused on a particular class of the Lévy models, that is the class of jump-diffusion models. This particular class obviously includes all the models we have seen so far, and allows the assets and other latent variables, such as volatilities or convenience yield, to exhibit sudden jumps.

Definition 2.10. *Let $Z = (Z_t)_{t\geq 0}$ be an adapted process, with $Z_0 = 0$ a.s. Then Z_t is a* ***Lévy process*** *if the following properties are satisfied*

i) Z has increments independent on the past, that is $Z_t - Z_s$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t$

ii) Z has stationary increments, that is $Z_t - Z_s$ has the same distribution as Z_{t-s} for all $0 \leq s < t$

iii) Z_t is continuous in probability, that is $\lim_{t\to s} Z_t \xrightarrow{\mathbb{P}} Z_s$.

Remark 2.1. *It is clear that a pure diffusive process driven by a Brownian Motion meets all the properties of Definition 2.10, then is a Lévy process.*

Looking at the market movements of the price, sudden and rare breaks can be found, at least in crises periods or economical booms. Such behaviours are due to the reactions of the markets to sudden critical informations available in the market during the time. Clearly, such conditions represents special situations and can be identified as rare events. Hence such a behaviour can be described well by a class of particular Lévy processes. In particular it can be modeled considering some point processes N that counts the occurrences of a rare and random event occured untill t, as

$$N_t = \sum_{n\geq 1} \mathbf{1}_{[\sigma_n,\infty)}(t) M_n$$

being σ_n a sequence of stopping times, representing the instant at which sudden situations occur, and M_n are random variables. Associated to such a process a term that takes into account how the news coming from the market act on the price of the asset is also needed. Such a term can be deterministic or stochastic as well. Usually the process N is assumed to be determined by a sum of independent Poisson processes, where each of them is assumed to have sudden jumps of constant size occuring at rare and not predictable intervals. The introduction of such terms allows us to take into account for sudden price movements caused, for instance, by default of some huge istitution such as the case of the crisis of 2008, or by governamental actions in financial markets or by situations of instability in the global policy.

We observe that the introduction of the jump terms is reflected in the stochastic differential equation assumed for the evolution of the asset price by the presence of a nonlocal term. Let us consider an n-dimensional risky asset X_t, taking values in $D \subseteq \mathbb{R}^d$. If $x_0 \in \mathcal{L}^2(D)$ and we allow X to have sudden jumps, then it is possible to describe the evolution of the X_t in its generality, by the following SDE

$$X_t = x_0 + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \int_D \gamma(z)m(ds, dz) \tag{2.9}$$

where $m(ds, dz)$ is the compensated jump martingale measure, under the risk-neutral probability measure we are considering of a Poisson process v adapted to $\mathcal{F}$. The coefficients μ and σ are respectively the drift vector and the diffusion matrix for the process X_t, and γ is the matrix of the jumps. For precise definitions and properties of random measures we refer to the works by Gihman-Skorohod in [37] and Jacod in [47]. We stress that it is possible to allow μ, σ and γ depend on the time t as well.

Also in such a case, for any choice of the parameters in the equation (2.9), a different market models is taken into account. However the presence of the jumps weakens the results of existence and uniqueness of the solution for the pricing problem related to the market, and the existence and uniqueness of the solution for the SDE (2.9). In particular, it is important to remark that, when we deal with stochastic differential equations, the problem (2.9) may admits two kinds of solutions X_t, that is a stochastic process verifying the equation (2.9).

Definition 2.11. *The problem (2.9) admits a* ***strong solution*** *if, for each Standard Brownian Motion* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, (W_t)_t, \mathbb{P})$*, there exists* X *such that* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, (X_t)_t, (W_t)_t, \mathbb{P})$ *satisfies (2.9).*

Definition 2.12. *The problem (2.9) admits a* ***weak solution*** *if* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, (W_t)_t, \mathbb{P})$ *is a Standard Brownian Motion and there exists* X *such that* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, (X_t)_t, (W_t)_t, \mathbb{P})$ *satisfies (2.9).*

We observe that in the case of the weak solutions, the probability space on which the solution is defined is not defined a priori, and it may depend on the initial data.

The presence of the jump terms has impact on the existence of the solution for the equation (2.9). In addition to the classical hypoteses on the Lipschitz continuity and sublinear growth for the coefficients μ and σ, some hypoteses on the nonlocal term have to be made.

In particular the coefficients are required to be globally Lipschitz continuous with sublinear growth, and the integral term is bounded. Such a result is given by Pham in [57], Section 2, and by Bodo, Thompson and Unny in [16], Section 4. The precise statement of the result can be given in the following

Theorem 2.1 (Strong existence, in Pham [57])**.** *Let X be the solution of the equation (2.9). Suppose that the coefficients μ, σ and γ and the measure $m(dz)$ verify the following conditions:*

i) The measure $m(dz)$ is positive σ-finite on $\mathbb{R}^d$, eventually with a singularity in 0, such that

$$\int_{|z|>1} m(dz) < +\infty,$$

ii) there exist $K > 0$, $c_0 \in \mathbb{R}$ and $\rho : \mathbb{R}^n \to \mathbb{R}_+$, with $\int_{\mathbb{R}^d} \rho^2(z)m(dz) < +\infty$,

iii) $|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq K|x - y|$ for each x, y in D,

iv) $|\gamma(x, z) - \gamma(y, z)| \leq \rho(z)|x - y|$ for each x, y in D,

then there exists a unique strong solution to the problem (2.9).

We observe that the Lipschitz condition *(iii)* and *(iv)* and the continuity of the coefficients imply the sublinear growth rate for the coefficients.

2.5 Arbitrage principle and pricing derivatives

When we deal with the market theory, a fundamental hypotesis is commonly made. This assumption comes directly from the rational behaviours of the actors in the market, and is the assumption the the *absence of arbitrage*.

Definition 2.13. *An **arbitrage** is a financial transaction such that*

i) no capital is committed at any time t,

ii) gives rise to earning with probability $P^e > 0$ and to a loss with probability $P^l = 0$.

From a mathematical point of view, the absence of arbitrage is perhaps the most important principle we take into consideration. If such an assumption is met, then there are never opportunities to make instantaneous risk-free profit. On the other hand, such a condition seems to be reasonable in the markets which are sufficently liquid. Indeed, what happens in the real world is that any presence of such opportunities takes place for a very short time, since, when such opportuinities exist, all traders in the market will follow a strategy making a risk-free profit. Such strategies move the prices of the products, and in a very short time such opportunities will be eliminated.

Definition 2.14 (Absence of arbitrage)**.** *Given a market $\mathcal{M}(X)$, if for all self financing strategies such that $\Delta_s \cdot X_s \leq 0$ for some s implies $\Delta_t \cdot X_t \leq 0$ for all $t > s$, then the market $\mathcal{M}(X)$ is said to be without arbitrage opportunities.*

From a mathematical point of view the absence of arbitrage means that there exists two equivalent probability measures $\mathbb{P}$ and $\mathbb{Q}$, that is each set $\mathbb{P}$-neglectable is $\mathbb{Q}$-neglectable as well. Such an equivalence is given in the next theorem, whose proof can be found in [24].

Theorem 2.2. *A market is without arbitrage opportuinities if and only if there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which the acualized assets' prices are martingales.*

The probability $\mathbb{Q}$ is usually called *risk neutral* or *equivalent martingale measure*. The latter definition has economical roots since the equivalent probability $\mathbb{Q}$ encloses the risk-adversion of the traders, and all the valuations under such a probability are exactly the one that would be given in a risk neutral world.

The absence of arbitrage opportunities, and then the existence of an equivalent probability measure, is very useful when we are focused on the problem of pricing contigent claims. In such a case, indeed, it is possible to get the fair price of the contract by following the so called *hedging strategy* we have defined in Definition 2.9 (see e.g. Proposition 1.14 in [3] or Proposition 1.2.7 in [51]).

Furthermore we can argue that the existence of a hedging strategy is referred to all the market and not just to a single derivative. Such a problem is related to the so called *completeness* of the market.

Definition 2.15. *A market is **complete** if all derivatives admits a hedging strategy.*

The assumption of the completeness of the market does not seem to answer to any economical or fundamental arguments. However, it represents a useful property of the considered market model. On the other hand, the absence of arbitrage opportunities is a fundamental assumption of the financial mathematics, and it is verified in the most markets. Thus a market model which drops such an assumption is not reasonable by the rational behaviour of the agents.

Furthemore an important theorem shows that the completeness of the market is ensured whenever the the market may be characterized by means of a unique equivalent martingale measure. This fact is granted if the coefficients of the equation (2.9) verify some mathematical requirements, that may be far from having direct “economical” consequences.

Theorem 2.3. *If the market $\mathcal{M}$ is without arbitrage opportunities, then it is also complete if and only if the equivalent martingale measure $\mathbb{Q}$ is unique.*

Such a result is widely known in financial mathematics. The proof of such a Theorem can be found, for instance, in [24].
However we remark that the completeness of the market may not hold true even in a very simple case, if we consider a market where more than one asset may be traded. In such a case, in fact, even if we assume that each asset is described by a pure diffusive equation, such as a standard Black and Scholes evolution, the no arbitrage hypoteses may automatically hold true, but, more assumptions are needed in order to guarantee the completeness of the market. In such a simple case, for instance, an additional hypotesys on the rank of the volatility matrix have to be done. Unfortunately, such an additional assumption does not seem to have any financial motivations.
If we consider any model more complicated with respect to the standard Black and Scholes' model, the uniqueness of the equivalent martingale measure can not hold well. This may happen for example when we consider stochastic volatility models or jump diffusion ones that introduce new sources of randomness with respect to ones considered in the standard Black and Scholes model.

The next result shows that, under suitable assumptions, the arbitrage price Y_t can be written as a deterministic smooth function u depending on the process X_t and the time t.

Proposition 2.1 (Corollary 1.17 in [3], Corollary 1.2.10 in [51])**.** *Let $\mathcal{M}(X_t)$ be the market without arbitrage opportunities and (T, ϕ) a contigent claim. We assume that there exists a hedging strategy Δ_t for (T, ϕ) and a deterministic continuous function $u(X_t, t)$ such that $u(X_t, t) = Y_t = \Delta_t \cdot X_t$ for all t. If u has the following regularities properties:*

$$u \in L^\infty(0, T; W^{2,\infty}_{pol}(D))$$
$$\partial_t u \in L^\infty(0, T; L^\infty_{pol}(D)),$$

then Y_t is exactly the arbitrage value of (T, ϕ).

The proof can be found for instance in [3] or in [51]. The result comes from the Itô's formula applied to the function $u(X_t, t)$ whenever we consider the process X_t as a free variable. Then, under the measure $\mathbb{Q}$, the quantity $Y_t = u(X_t, t)$ is a martingale, and the market $\mathcal{M}(X) \cup \mathcal{M}_0(Y)$ is without possibility of arbitrage. Then the thesis comes from Proposition 1.14 in [3].

If the equation (2.9) is linear the Δ-hedging technique suggests u as the solution of a final value problem. In the case the (2.9) is a pure diffuve process, by the Itô's calculus it can be proven that u has to be the solution of a linear determinisitic partial differential equation, if it is a jump diffusion process, then the function u has to verify a deterministic integro-partial differential problem. Hence, the well-posedness of the pricing problem can be addressed by studing existence and uniquenss of the solution to a given PIDE, where, the integro-differential operator is determined once the market is chosen, and the final condition is given by the specific considered contigent claim.

As a consequence of Proposition 2.1, once the existence of a hedging strategy is proven, the arbitrage price of a contract can be determined for each time t, starting from such a hedging strategy.

On the other hand, the existence of a hedging strategy can be shown once some assumptions on the coefficients of (2.9) are put in place. Such coefficients are required to ensure the existence of an equivalent martingale measure, hence the absence of arbitrage. We recall the result of Theorem 10.9 in [9].

Proposition 2.2 (Theorem 10.9, in [9])**.** *Consider the pure diffusive problem (2.9) where the coefficients μ and σ are bounded and Lipschitz continuous. Suppose furthermore the matrix $\sigma(x)\sigma^\top(x)$ be elliptic. Let (T, ϕ) a European-type derivative such that $\mathbb{E}(\phi^q) < \infty$ for some $q > 1$. Then there exists y_{min} such that for each hedging portfolio Y for ϕ, it holds $Y_s \geq y_{min}$. Furthermore there exists a hedging strategy Δ_t such that $Y_s = \Delta_s \cdot X_s = y_{min}$.*

We observe that the assumptions on the coefficients are needed in order to guarantee the absence of arbitrage.

Until now we have spoken about the stochastic differential problems, however in force of Proposition 2.1 the pricing by arbitrage may be addressed by solving a determinisitic differential problem, by means of *virtual hedging* and Itô's calculus. Furthermore, if the market is complete, for any contigent claim, traded in the market, we get an unique and well defined arbitrage value simply by applying such a technique. On the contrary, if the completeness is not met, then the value by arbitrage is not uniquely defined.

In this case, indeed, the Δ-hedging approach is still valid, but it is not able to give us the unique arbitrage value. In this case, the value by arbitrage is not uniquely defined, and the pricing by hedging provides a class of deterministic partial differential operators, depending on a parameter which stands for the *price of risk*. Which is the arbitrage value of a given function depends on the particular choice for the price of risk. From an economical point of view, such a quantity corresponds to a particular choice of the hedging strategy followed by the traders, that may want to minimize his risk, or maximize his expected returns. From a mathematical point of view, different choices of price of risk correspond to different choices of the equivalent martingale measure. Hence, different strategies may be followed. However, also in this case it is possible to choose a particular price of risk and study the corresponding equation, in order to not loose the powerful tool of differential approach.

For the case of the Black and Scholes market, as a standard result it can be proven that it is without arbitrage opportunities and it is complete as well. In particular, combining the results given in Proposition 1.20 and 1.21 in [3] we can state

Proposition 2.3. *The Black and Scholes' market is without arbitrage opportunities. Furthermore it is complete, and the arbitrage price of the European contigent claim* (T, ϕ) *is given by the solution of the final value problem on* $(0, \infty) \times (0, T)$*:*

$$\begin{aligned} &\partial_t u + \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} u + rS\partial_S u - ru = 0 \\ &u(S,T) = \phi(S) \end{aligned} \tag{2.10}$$

and the hedging strategy is given deterministically as a function of (S_t, t) *by*

$$\begin{aligned} &\Delta^0_t = e^{-rt}\big[u(S_t,t) - S_t\partial_S u(S_t,t)\big] \\ &\Delta_t = \partial_S u(S_t,t) \end{aligned} \tag{2.11}$$

The proof is given in [3]. The result of the absence of arbitrage is achieved providing directly an equivalent probability measure $\mathbb{Q}$ under which the actualized stock prices are martingale. In order to achieve such a result the Girsanov Theorem plays a crucial role.

The completeness of the market is achieved observing that, given a hedging strategy Δ, the value of the portfolio is given by

$$Y_t = e^{rt} S^0_t \Delta^0_t + \Delta_t \cdot S_t.$$

Hence Y_T has to verify

$$Y_T = \phi(S_T) + \int_t^T \left[rY_u - \Delta_u S_u(\mu - r)\right] du - \int_t^T \Delta_u \sigma dW_u. \tag{2.12}$$

The thesis comes from the application of the Itô's lemma to the function $u(X_t, t)$

$$u(S_t, t) = \phi(S_T) - \int_t^T \partial_t u + \frac{1}{2}\sigma^2 \partial_{SS}^2 u + \mu S \partial_x u d\tau - \int_t^T \sigma S \partial_S u dW_\tau \tag{2.13}$$

and identifying each term of the equations (2.12) and (2.13). Hence we get both the expression of the hedging strategy and the value by arbitrage of the derivative ϕ.

Chapter 3

Recent approaches to the valuation equation

We have seen in the previous chapter that given a particular market, the problem of pricing a contigent claim can be addressed by finding the expected value of the final payoff, once the stochastic process X, solution of the stochastic differential equation (2.9), is available. Indeed, if such a solution exists, and the expected value of the final payoff is well-defined, then, it represents the fair value of the contract. Furthermore, if there exists a hedging strategy, the expectation of the final payoff can be found applying arbitrage arguments. Another way that can be followed when one deals with the pricing of a contigent claim, is represented by Itô's lemma. Indeed, when the evolution of the the state space X is given, in its general form, by the stochastic differential equation (2.9), then it is possible to find the arbitrage price by solving the Cauchy problem of a PIDE coming from the Itô's lemma, with a given final value. The particular form of the integro-differential operator $\mathcal{A}$ will depend on the particular choice of the market, and the final value will depend on the particular considered contigent claim.

In last decades, several works have been made, focused on the solution of the PIDE. In particular, the topic that has been mainly studied is the existence and uniqueness of the solution of a given PIDE, and the regularity of such a solution. These results obvioulsy depend on the particular form of the integro-differential operator, the particular Cauchy problem considered, and in which *sense* such solutions are looked for. As an example, when we are dealing with pure diffusive operators, and the diffusive matrix is uniformly non singular in all the state space where the process is defined, then the solution could be found making use of the Sobolev theory and standard theorems of Lévy generators. In such conditions, may exists a solution in a *strong sense*, that is also regular (see for instance [28] and the reference therein for a discussion on this topic). Unfortunately, many problems in finance do not fit such conditions, and a solution in a weaker sense have to be found.

A powerfull approach is represented by viscosity solution theory that allows to deal with singular diffusive matrices, also in presence of non local term in the operator. The drawback of such a theory is that only the continuity of the solution is granted when the initial data are continuous.

This Chapter is organized as follows. The first section is devoted to indicate the main problems of interest in finance that cannot be dealt with the present literature. In particular the problems we are faced on are indicated, and we point our attention in understanding which kind of problems cannot be dealt with the present literature. Furthermore, in the last section we give the definitions that allow us to use the viscosity solution approach to solve pricing problem, and is an outlook on the *state of the art* in pricing problems.

3.1 Approaches for the evaluation equation

As we have seen, when the market fits the class of the affine model and the final payoff is affine or exponential-affine, then a solution can be found by solving a system of ordinary differential equations in the Fourier space, and then inverting the solution coming back to the state space. On the other hand some results can be obtained if the prices are assumed to evolve following some special model that are quite handy from a mathematical point of view. Some of the most popular are Geometric Brownian Motion, or the Heston model in some simple cases. As we have already stated such models are not able to explain all the features exhibited by the products traded in the market. Unfortunately, when we consider the general case for European derivatives, the valuation problem have to be dealt with by solving the corresponding Partial Integro-Differential Equation (PIDE), the general form of which is given by

$$\begin{aligned} &\partial_t u(x,t) + \mathcal{A}u(x,t) - c(x)u(x,t) = f(x,t) \qquad && (x,t) \in D \times (0,T) \qquad (3.1)\\ &u(x,T) = \phi(x) && x \in D \end{aligned}$$

where D is a (possible unbounded) subset of $\mathbb{R}$. The link between the solution of a problem of the type (3.1) and the fair value of the corresponding derivative comes from the representation formula. The integro-differential operator $\mathcal{A}$ can be divided in two terms $\mathcal{A}_d$ and $\mathcal{J}$, the former being a pure integral operator, the latter a pure diffusive operator. In the rest of the work we assume the following expression for the operator $\mathcal{A}$

$$\begin{aligned} \mathcal{A}g(x) &= \mathcal{A}_d g(x) + \mathcal{J}g(x) \\ &= \langle \nabla g(x), \mu(x) \rangle + \frac{1}{2} tr\left(\nabla^2 g(x) a(x)\right) \\ &+ \int_D [g(z) - g(x)]\, m(x,dz). \end{aligned} \qquad (3.2)$$

where we have indicated by

$$\mathcal{A}_d g(x) = \langle \nabla g(x), \mu(x) \rangle + \frac{1}{2} tr \left(\nabla^2 g(x) a(x) \right) \quad (3.3)$$

$$\mathcal{J} g(x) = \int_D \left[g(z) - g(x) \right] m(x, dz). \quad (3.4)$$

The matrix $a(x) = \sigma(x)\sigma(x)^\top$ is the diffusion matrix of the state process X_t, μ is the drift under some risk-neutral probability measure, c is the discount rate function. Such quantities are characteristic of the special model assumed for the process X. The terms ϕ and f, instead, are specific for the particular considered problem and represent respectivelly the final payoff at time T, and a continuous yield or a running cost. The presence of sudden jumps in the evolution of the process X_t is taken into account by the integral term $\mathcal{J}$. The measure m includes the jump intensity and the probability distribution of X_t after the jump as well, under the considered risk-neutral probability measure.

We notice that, if the measure $m(x, dz) = \delta(x)dz$, then the operator $\mathcal{A}$ reduces to the pure diffusive case.

On the other hand, even if the state variables are assumed to evolve following an affine process, but the final payoff cannot be expressed in an affine or exponential-affine form we have to deal with the general problem given by (3.1). This is often the case when the Arithmetic Asian option is considered.

An Arithmetic Asian option is a contingent claim in the sense of Definition 2.6 where the average of the underlying on a fixed (or floating) time interval is used in order to determine the final payoff. As an instance, in the case of a European Arithmetic Asian call option, the final payoff may be given by

$$\left(\frac{1}{T} \int_0^T S_t dt - K \right)_+, \qquad \text{or} \qquad \left(\frac{1}{T} \int_0^T S_t dt - S_T \right)_+.$$

When we deal with the Arithmetic Asian option, it is possible to consider the integral term as an independet variable, and solve the Cauchy problem of the form given by (3.1). The new problem lies in a bidimensional space. Indeed, in such cases then we can introduce an additional stochastic term Z_t that solves the differential equation

$$dZ_t = S_t dt \quad (3.5)$$

and an analogue form with the logarithm of S_t, if we are dealing with Geometric Asian options (see for example [58]). Then if we consider the process (S_t, Z_t), the final payoff

$\phi(S_t)$, given in the new bidimensional problem becomes, for a fixed time T,

$$\phi(S_t, Z_t) = \left(S_T - \frac{1}{T}Z_T\right)_+ . \tag{3.6}$$

The particular form of the operator $\mathcal{A}$ obviously depends on the model of the market, however, given the expression of (3.5), the diffusion matrix $a(S_t, Z_t)$ is degenerate in a subspace of D.

In the case the underlying product is assumed to follow a Black and Scholes process then it is possible to reduce the number of the dimensions of the considered problem (see for example section 3.3.4 in Zhu, Wu, Chern [65]). The idea that degenerate diffusions can be reduced to lower-dimensional nondegenerate diffusions on a submanifold of the underlying asset space was carried on by Barraquand and Pudet in [11]. Unfortunately, in general, the dimensional reduction is possible only under suitable homogeneity properties, and this is not the case when we assume that the underlyings evolve following a process, that is more realistic with respect to Black and Scholes one, such as the Heston model.

We observe that when the reduction of the dimensions cannot be put in place, the Asian option introduces a degenerate integro-partial differential problem for which the standard theory of Sobolev space cannot be applied. The classical theory of Sobolev spaces ensures large regularity to the solution. Unfortunately, this theory cannot be applied when the diffusion matrix $a(x)$ is not positive defined in all the state space D, as it happens in this case.
As an instance, for the asian options, in the case the underlying product is assumed to evolve following the Heston model, the state space is given by (S_t, V_t, Z_t) and the integro-differential operator $\mathcal{A}$ is purely diffusive $\mathcal{A}_d$, and is determined by

$$\mathcal{A}_d u(t,s,v,z) = rs\partial_s u + \kappa(\theta - v)\partial_v u + s\partial_z u + \frac{vs^2}{2}\partial_s^2 u + \frac{\sigma_0^2}{2}v\partial_v^2 g + \rho sv\partial_{sv}^2 u. \tag{3.7}$$

The diffusion matrix $a(x)$ is then given by

$$a(x) = \frac{1}{2}\begin{pmatrix} vs^2 & 2\rho sv & 0 \\ 2\rho sv & \sigma_0^2 v & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This very simple example shows that, the standard theory of Sobolev spaces is not able to deal with such kind of problems. We observe that this particular kind of derivative is one of the most handy from a mathematical point of view, and is also one of the most common products traded in the market, expecially in the commodity ones. Furthermore, given the fact that the Black and Scholes model is not able to deal with the behaviour of the assets in the markets, in practical situations, the dimensional reduction is not available.

On the other hand, if we are interested just in valuing the derivatives, taking for granted the existence and uniqueness of the solution, such kind of problem could be solved by using Monte Carlo simulation technique, where the fair price of such a contract, is defined, in the risk-neutral world, by the expected value of the contract. However, we have to remark that pricing derivatives by Monte Carlo simulations has several drawbacks. Some of them are practical, such as the fact that the time of the execution may become exponentially high when the number of the underlying increases. On the other hand, some theoretical problems can arise as well. Indeed, in order to guarantee high precision to the evaluation, the time interval of the simulation have to be very small. In such a case, the discretization procedure, does not always guarantee that the distribution obtained in the Monte Carlo simulation is correct, as shown in [6] where the Heston model is considered. As a final consideration we observe that when we deal with coefficients that are fast growing at infinity, as could be the case in some stochastic volatility models, the uniqueness of the solution of the Cauchy problem (3.1) may not be hold and the expectated value of the final payoff may be not defined. In such conditions, the Monte Carlo simulation does not converge and the results strongly depend on the number of simulations that have been put in place.

For these reasons, it is important to know *a priori* if the existence and uniqueness of the solution of the pricing problem are granted. In this case, several numerical techniques are available. Furthermore, it is well-known from numerical analysis that when the solution $u(x,t)$ is regular enough, then numerical procedures converge fastly to the analytical solution. Such a remark suggests that also the study of the regularity of the solution represents one important topic to be improved.

In order to perfom such kind of analysis, a very powerful tool is represented by viscosity solution theory. In this framework it is possible to look for the solutions to the integro-differential problem (3.1) even in the space of continuous functions since they are allowed to be not differentiable. This kind of solutions has to be intended in a weak sense, but such a theory allow us to deal with singular diffusion matrices, such as the considered cases. Furthermore the viscosity solution is well suited to be computed numerically as shown in [10] (where a discretization procedure is shown) and in [31]. Unfortunately, as we have already stated, the drawback of this theory is that the solution is ensured to be only continuous if data are continuous but, for the general case, it is allowed to be not differentiable. However, when it is proven that the viscosity solution admits some kind of additional regularity, and is smooth enough, then, the numerical procedures fastly increase their rate of convergence towards the analytical solution. This is the reason why understanding the regularity of such solutions is of crucial importance both from a speculative point of view and for practictioners.

On the other hand, it is possible to show that, if the pricing problem admits both a viscosity solution, and a solution in a stronger sense, then they may coincide. This fact is shown as an instance in [57], and in [22]. The former work deals with a special case of a degenerate diffusive hypoelliptic Kolmogorov operator, the latter deals with a classical solution to the evaluation problem where the integro-differential operator has positive defined diffusive matrix and Lipschitz conditions on the coeffcients. Such results are true once the uniqueness of the solutions is proven.
On the other hand, it could be possible to imagine that if the problem admits a unique viscosity solution $u(x,t)$ in a given domain D, and admits a unique $v(x,t)$, solution in a stronger sense in some domain $K \subset D$ such that $v(x,t) \equiv u(x,t)$ for each $x \in \partial_{\mathcal{P}} K$ then the solutions $u(x,t)$ and $v(x,t)$ have to coincide in K. Unfortunately, this fact is not obvious, since the viscosity solution $u(x,t)$ may not coincide with the viscosity solution $u_K(x,t)$ for the problem restricted in K.

3.2 Recent developments on finance mathematics

The viscosity solution theory has been originally developed in the most celebrated work [19] provided by Crandall, Ishii and Lions. In this work the authors deal with nonlinear second order Partial Differential Equations, where no integral terms are considered. It is clear that if it is possible to define, in some ways, the derivatives even for functions that are merely continuous, the space of the solutions for a given problem enlarges, and then it is possible that a problem that does not admit a solution in a classical sense, admits a solution in the viscosty framework.

On the other hand, the main important feature in the viscosity solution theory is exactly the possibility to define, in some ways, the derivatives in a weakr sense. In particular, the theory of viscosity solutions has its main point in the definition of the so-called *parabolic semijet* which allows the definition of the derivative in the viscose sense of a given function u even if it is merely continuous. It is possible to give two equivalent definitions of the parabolic semijet, one is local, taking into account the behaviour of the function near a given point, the other is global. Following [17] we give

Definition 3.1. *Given u upper semicontinuous function ($\mathcal{USC}(D \times [0,T])$), the* ***parabolic super 2-jet*** *of u at the point $(\bar{x},\bar{t}) \in D \times (0,T)$ is the set $\mathcal{P}^{2,+}u(\bar{x},\bar{t})$ such that the following condition is verified*

$$\begin{aligned} \mathcal{P}^{2,+}u(\bar{x},\bar{t}) &= \{(\partial_t g(\bar{x},\bar{t}), \nabla g(\bar{x},\bar{t}), \nabla^2 g(\bar{x},\bar{t})) : g \in \mathcal{C}^{2,1}(D, \times[0,T]), \text{ and} \\ & \quad (u-g)(x,t) \le (u-g)(\bar{x},\bar{t}) = 0, \quad \forall (x,t) \in D \times [0,T]\}. \end{aligned} \tag{3.8}$$

We will say that a function g as above is a ***test function*** *for $\mathcal{P}^{2,+}u$ at $(\bar{x},\bar{t})$. For a function*

*u lower semicontinuous ($\mathcal{LSC}(D \times [0,T])$) the **parabolic lower 2-jet** is defined as the set of points such that $\mathcal{P}^{2,-}u = -\mathcal{P}^{2,+}(-u)$.*

Remark 3.1. *The definition 3.1 is equivalent to require that there exists a function g such that $u - g$ has a strict global maximum at $(\bar{x},\bar{t})$, that is*

$$(u-g)(x,t) < (u-g)(\bar{x},\bar{t}) = 0 \tag{3.9}$$

for each $(x,t) \in D \times [0,T]$ and $(x,t) \neq (\bar{x},\bar{t})$.

Definition 3.2. *We call a **good test function** for $\mathcal{P}^{2,+}u$ at $(\bar{x},\bar{t})$ (resp. $\mathcal{P}^{2,-}u$) at (x,t) any function $g \in \mathcal{C}(\mathbb{R}^d \times [0,T))$ such that $u - g$ has a strict global maximum (resp. minimum) at (x,t) and $u(x,t) = g(x,t)$.*

In particular, in the viscosity solution framework, the derivatives of the differential operator are not applied to the function $u(x,t)$ that is just continuous, but to the good test functions g. The notion of the viscosity solution for pure differential operators $\mathcal{A}_d$ is then given as

Definition 3.3. *A function $u \in \mathcal{USC}(D \times [0,T])$ (resp. $u \in \mathcal{LSC}(D \times [0,T])$) on $D \times [0,T]$ is a **viscosity subsolution** (resp. **supersolution**) of (3.1) if*

i) for every $(x,t) \in D \times (0,T)$ and any test function g for $\mathcal{P}^{2,+}u$ at (x,t) (resp. $\mathcal{P}^{2,-}u$), it holds

$$\partial_t g(x,t) + \mathcal{A}_d g(x,t) - c(x)u(x,t) \geq f(x,t) \qquad \textit{(resp. } \leq\textit{)} \tag{3.10}$$

ii) $u(x,T) \leq \phi(x)$, (resp. $\geq$) for all x in D.

*A function that is both a viscosity subsolution and a viscosity supersolution of (3.1) is a **viscosity solution** for the problem (3.1).*

As in the case of integro-partial differential equations, it is possible to give a notion of derivative in the viscose sense also in the case of ordinary differential equations, through the introduction of the super-jet.

Definition 3.4. *Given $u \in \mathcal{USC}([0,T])$, the **super 1-jet** of u at the point $\bar{t} \in (0,T)$ is the set*

$$\mathcal{P}^{1,+}u(t) = \{g'(t) \in \mathcal{C}^1([0,T]), (u-g)(t) \leq (u-g)(\bar{t}) = 0, \forall t \in [0,T]\}. \tag{3.11}$$

*We will say that a function g above is a **test function** for $\mathcal{P}^{1,+}u$ at $\bar{t}$.*

Also in this case we can assume that $(u-g)(t) < (g-u)(\bar{t}) = 0$ for all $t \in [0,T]$ and $t \neq \bar{t}$. This type of function is still called a **good test function** for $\mathcal{P}^{1,+}u$ at $\bar{t}$.

Definition 3.5. *Let $\Gamma : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. An I-valued function $v \in \mathcal{USC}([0,T])$ is a **viscosity subsolution** of*

$$v'(t) + \Gamma(v(t)) = 0 \tag{3.12}$$

for $t \in (0,T)$, if for every $t \in (0,T)$ and for any test function g for $\mathcal{P}^{1,+}v$ at t, it holds,

$$g'(t) + \Gamma(v(t)) \geq 0. \tag{3.13}$$

The theory of viscosity solutions has been later extended to Partial Integro-Differential Equations. In this case the notion of the parabolic semijet is still valid, since it takes into account the differential part of the operator $\mathcal{A}$. The notion of the viscosity sub/super-solution has to be modified instead. In particular the new nonlocal term has to be taken into account. Then in the relation (3.10) the operator is the full integro-differential operator $\mathcal{A} = \mathcal{A}_d + \mathcal{J}$ and the integral is applied to the function $u(x,t)$. Then the solution $u(x,t)$ is required to be integrable with respect the measure $m(x,dz)$.
For a complete tratment of these arguments we refer mainly to the work [19], [32], [44], [45] and the improvements on this theory some of them being [2], [57], [13], [48], [49], [5], [1], and the references therein.

Despite the viscosity solution theory enlarges the space of the solutions and allows to deal with singular jump-diffusion problems, unfortunately, in the general case given by the problem (3.1), the modern literature is not always able to ensure the existence, uniqueness and regularity of the solution of the problem (3.1). In particular, several problems arise when some conditions are verified. In the most cases, such conditions are referred to the coefficients of the operator rather than the specific final payoff, hence they are quite general for a given class of models.

In particular some problems can occur when the diffusion matrix $a(x)$ is singular on the boundary, or is identically zero in some directions. As we have seen, the former condition is verified in pricing Asian options when some stochastic volatility models are considered, such as Heston model. The latter is the case of Asian options, of some path dependent volatility models, like Hobson-Rogers model, or models where some components are pure jump. We observe furthermore that at the boundary of the domain D, in the case of pricing Asian options under Heston model, the rank of the diffusion matrix $a(x)$ reduces and it becomes identically zero in the subspace $(s,v,z) = (0,0,z)$.
Other problems comes when the drift μ and the matrix σ are fast growing near the boundary or at infinity, or they are not Lipschitz continuous up to the boundary of the domain D, such as the models with "square root" diffusion for some components, like CIR or Heston

model. On the other hand, it is well-known that when the coefficients are uniformly Lipschitz continuous, the stochastic process X solution of the problem (2.9) is strongly regular with respect to initial data and such a regularity could be directly inhered by the solution u of the valuation problem (3.1). Unfortunately, when the coefficients loose their regularity near the boundary this property is definetly lost.
We observe furthermore that, in some cases, the pricing problem is formulated on a state space D which has a boundary, but no boundary conditions are specified. This is often the case when D has a boundary, or when the problem defined on $\mathbb{R}^d$ is collapsed in a bounded domain (we refer to [65] for an example on this technique). Reguarding this particular case, we remark that several works have been made in the last years that conjecture that in most cases of interest, boundary conditions are not really needed, and they are redundant from a mathematical point of view (see for example [39] and [29]). For such a reason, as stated in [29], it is common to speak about the *behaviour near the boundary* instead of *boundary conditions*. On the other hand, we see in Chapter 4, that in some cases the boundary of the domain D is prohibited to the process X, as shown in [17], and speaking about behaviour near the boundary is not only appropriate but is correct from a rigorous mathematical point of view. We observe that in [17] is proven that such a feature is met when the coefficients μ, σ and the measure m verify very general hypoteses, and a suitable Lyapunov type condition is hold true. This fact has very important implications on the existence and uniqueness of the problem (3.1).

Although there are many results on viscosity solutions, the existing ones are often not sufficient to deal with the above described features, even in the linear case. Obviously, the problem of the regularity of the solution is neither dealt with the present literature.

In particular, Pham in [57] proves the existence and uniqueness of the viscosity solution to the problem (3.1) when the integral term is included. In his work, it is assumed that the coefficients are globally Lipschitz continuous, and have at most sublinear growth. Under these assumptions the author is able to prove also some estimates on the dependence on the initial data. Furthemore in this work it is proven that, roughly speaking, if the diffusion matrix is uniform positive defined on D and the final payoff is Hölder continuous, then the existence of a unique classical solution is granted and it coincides with the viscosity one. Unfortunately such assumptions seem to be quite strong and are not verified in many cases we have already seen. Alibaud in [1] proves existence and uniqueness of viscosity solutions for PDE's with nonlocal terms with bounded intensity for the jumps. Furthemore, the coefficients are assumed to be uniformly continuous.

Other works are presented by Barucci, Polidoro and Vespri in [12], Di Francesco, Foschi and Pascucci in [21] and Pascucci in [56] where a study of the evaluation equation with a special structure is provided. In particular, the operator considered in the valuation equation by the authors is pure diffusive, and the diffusion matrix $a(x)$ is uniformly positive defined on a linear subspace of $\mathbb{R}^d$ with dimension $m < d$ (see, for example, Assumption 3.2 in Pascucci [56]). These works make use of the results coming from the hypoelliptic operators theory. When the operator is hypoelliptic, if the sourcing term is smooth and the final value is a continuous function, then each solution in the sense of distribution is a solution in the classical sense as well. We recall that the Kolmogorov operator considered by authors is of the form

$$\mathcal{A} = \sum_{i,k=1}^{m} a_{ik}(x,t)\partial^2_{ik} + \sum_{k=1}^{m} b_k(x,t)\partial_k + \sum_{i,k=1}^{d} b_{ik}\partial_k + \partial_t. \tag{3.14}$$

The dependence on the time derivatives can be inverted if we consider a time-reverted problem.

A very important result for the class of Kolmogorov operators of the type (3.14), was pointed out in 1976 by Hörmander in his work [42]. In such a paper, the author gives sufficient conditions under which the operator $\mathcal{A}$ is hypoelliptic, whenever the coefficients are smooth functions. Such conditions are almost sufficient too. In particular it has been shown that if the rank of the Lie algebra generated by the vector fields $\partial_1, \ldots, \partial_d$ and Y defined by

$$Y := \langle Bx, \nabla\rangle + \partial_t$$

is equal to $d+1$, where B is the constant matrix of the operator $\mathcal{A}$, then the operator $\mathcal{A}$ has sufficient regularizing effect, and the operator is hypoelliptic. In his work Hörmander takes into account all the commutator of the vector fields. Such a condition takes often the name of the rank Kalman's condition. The importance of this work is that, for the first time it is pointed that the smoothing effect is granted not only by the diffusive components present in the operator $\mathcal{A}$, but also by all their commutators, and their commutators with Y.

It is well-known that the natural framework for the study of the operators satisfying the Hörmander condition is the analysis of Lie group, since the works by Folland [33], Rothschild and Stein [59]. On the other hand all the results given by Lanconelli, Pascucci, Polidoro, are in the Lie group structure related to Kolmogorov operators. An explicit expression of the group law is defined by

$$(\xi, \tau) \circ (x, t) = (x + E(t)\xi, t + \tau)$$

where $E(t) = exp(-tB)$ and B is the constant matrix of the operator $\mathcal{A}$. In their works the authors show all the results not in the usual norm, but in the norm that seems to be more appropriate in the considered case. Such a norm takes into account the presence of the term

B, in particular, for each $\alpha \in (0,1)$ the following norms are defined

$$||u||_{C_B^\alpha(\Omega)} = \sup_\Omega |u| + \sup_{(x,t)\neq(\xi,\tau) \text{ and } (x,t),(\xi,\tau)\in\Omega} \frac{|u(x,t)-u(\xi,\tau)|}{||(\xi,\tau)^{-1}\circ(x,t)||^\alpha} \quad (3.15)$$

$$||u||_{C_B^{1,\alpha}(\Omega)} = ||u||_{C_B^\alpha(\Omega)} + \sum_{k=1}^m ||\partial_i u||_{C_B^\alpha(\Omega)} \quad (3.16)$$

$$||u||_{C_B^{2,\alpha}(\Omega)} = ||u||_{C_B^{1,\alpha}(\Omega)} + \sum_{i,k=1}^m ||\partial_{ik}^2 u||_{C_B^\alpha(\Omega)} + ||Yu||_{C_B^\alpha(\Omega)}. \quad (3.17)$$

As pointed by the authors, any $u \in C_B^\alpha(\Omega)$ is Hölder continuous in the usual sense since

$$||(\xi,\tau)^{-1}\circ(x,t)|| \leq c|(x,t)-(\xi,\tau)|^{\frac{1}{2r+1}}$$

where r is integer and depends on the matrix B. Then, the following space can be defined for each $p \geq 1$

$$S^p(\Omega) = \{u \in L^p(\Omega) : \partial_i u, \partial_{ik} u, Yu \in L^p, i,k = 1,\dots,m\}$$

and the related norm

$$||u||_{S^p(\Omega)} = ||u||_{L^p(\Omega)} + \sum_{k=1}^m ||\partial_k u||_{L^p(\Omega)} + \sum_{i,k=1}^m ||\partial_{ik}^2 u||_{L^p(\Omega)} + ||Yu||_{L^p(\Omega)}.$$

Then, most of the results obtained by Baruci, Polidoro, Vespri, Lanconelli and Pascucci takes into account the solutions u that are defined in the space S^p. This result implies a further regularity for u and not just the continuity. On the other hand, the problem studied in these works rules out many cases of interest for financial applications. In particular no sudden jumps can be included.

Pascucci in [56] shows that no boundary conditions are needed when we are dealing with a pricing problem where the differential operator is of the type (3.14). In this work the author proves the existence and uniqueness of a strong solution to the ostacle problem in the framework we have seen so far. We remark that the differential operator of many models used in mathematical finance can be expressed as (3.14). Unfortunately the additional requirement of a uniformly positive defined diffusive matrix $a(x)$ in a linear subspace is not met in several models , such as Heston model where the diffusion matrix $a(x)$ vanishes near the boundary. Furthermore, the presence of the jumps is absolutely excluded in such a framework.

The study of the existence and uniqueness in the case of the correlated interest rate model proposed by Bernaschi in [14] is included in the work provided by Amadori in [5]. Indeed, in the pure diffusive case she assumes that the coefficients μ and σ are uniformly Lipschitz continuous away from the boundary. On the other hand near the boundary they are allowed to loose their global Lipschitz continuity, but they have to satisfy a limit condition that ensures that the process is regular enough. In particular, in dimension 1 the coeffcients are required to verify

$$0 < \frac{1}{2}\sigma^2(x) \leq \frac{x}{1-cx}b$$

for some $c > 0$. As it is clear this condition generalizes the well known Feller condition. On the other hand the case of the Heston model is not included in such a framework, and even the case of the of the Asian options as pointed in [17].

In the framework of classical solutions Ekström and Tysk in [29] prove the existence and unqueness of a classical solution to the pricing problem (3.1) when the price of the asset is assumed to evolve following a Black-Scholes like model, and the volatility is allowed to be a stochastic process. We observe that the problem considered by the authors is general enough to include the Heston model. Unfortunately, some additional assumptions on the final payoff are required. In their work, the authors provide some important considerations dealing with existence and uniqueness of the solution when the process S_t and the volatility V_t are positively correlated. In this work the authors addresses the issue of lakness of boundary conditions by adding appropriate ones. However, if the stochastic process does not reach the boundary of the domian, such conditions are not really needed, as stated by authors themselves, that argue that if it is the case, it is more appropriate to speak about the behaviour near the boundary rather than boundary conditions.

In the next chapters we overcome such results allowing the processes to have sudden jumps. Furthermore, we suppose furthermore general conditions on the coefficients for the integro-differential operator $\mathcal{A}$.

Chapter 4

Regularity of degenerate processes

During the years, several works have been presented in order to understand both the existence and regularity of the valuation equation of the type (3.1). Furthermore, a number of efficient numerical methods has been proposed in order to find a numerical solution. Most of these techniques are developed taking for granted the existence and regularity of a suitable solution for the considered problem. Althought this fact, often, the PIDE related to many models of interest in modern financial mathematics exhibit very stiff features, such as singular diffusion matrices and coefficients that are only locally Lipschitz continuous up to the boundary. Furthermore, in general, boundary conditions are not specified. In these cases no results of existence and uniqueness of the solution to the problem (2.9) is ensured. A forthcoming work provided by Costantini *et al.* on Finance and Stochastics (see [17]) proves the well-posedness of the martingale problem under very general assumptions on the integro-differential operator $\mathcal{A}$, provided that a Lyapunov-type condition is satisfied.

The Chapter is organized as follows. The first section is devoted to give a very brief introduction to the semigroup theory, expecially applied to the martingale problem, highlighting the links between the semigroup approach and the Markov processes. In the second section we give the main assumptions that will hold true for the sequel. Furthermore we report one of the main results provided in [17]. In particular, under general conditions on the operator $\mathcal{A}$ and the existence of a Lyapunov-type condition, the well-posedness of the martingale problem $(\mathcal{A}, P_0)$ for any initial distribution P_0 is shown. In the last section the assumptions made in [17] are still valid, but we restrict ourselves to consider pure diffusive processes. Under such conditions, some estimates of continuity with respect to the initial data for the process X are provided.

All the results of this chapter can be applied, for instance, to Asian option pricing, that is a very common instrument in the energy markets, pricing in path dependent volatility models and in jump diffusion stochastic volatility models. We verify our Lyapunov type condition in several examples, among which the Arithmetric Asian option in the Heston model.

4.1 An outlook on semigroup approach to martingale problem

In this section we briefly recall some basic definitions and results provided in [28], in the framework of the semigroup theory and the martingale problem approach. Such an approach represents a very powerful tool when we deal with stochastic processes. Indeed, as shown in [28] it is possible to find several properties of the processes and the quantities related to them by studying a suitable problem driven by an integro-differential operator. Such results are strongly used in [17]. In particular, since the operator semigroup theory provides a primary tool for the study of Markov processes, such an approach allows us to deal with the martingale problem for a given infinitesimal generator $\mathcal{A}$, in a very powerful framework. For a more detailed discussion to semigroup theory and its implication to the stochastic processes and the main results on the existence of the martingale problem we refer to [28] and the references therein.

Definition 4.1. *A one-parameter family $\{T(t) : t \geq 0\}$ of bounded linear operators on a Banach space E is called semigroup if the following properties are met:*

i) $T(0) = \mathbf{1}$

ii) $T(s+t) = T(s)T(t)$ *for all* $s, t \geq 0$.

where **1** is the identity operator.

Definition 4.2. *A semigroup $\{T(t)\}$ on E is said to be **strongly continuous** if for every $f \in E$ we have*

$$\lim_{t\to 0} T(t)f = f.$$

As an instance, if we consider a bounded linear operator on E whose representation is given by the matrix B, if we define, for $t \geq 0$

$$e^{Bt} = \sum_{k=0}^{\infty} \frac{B^k t^k}{k!} \tag{4.1}$$

then a direct calculation shows that $\left\{e^{Bt}\right\}$ verifies the properties of semigroup given in the Definition 4.1. Furthermore, it is easy to show also that it is strongly continuous.

Definition 4.3. *A semigroup $\{T(t) : t \geq 0\}$ is a **contraction** semigroup if $||T(t)|| \leq 1$.*

Proposition 4.1. *Let $\{T(t)\}$ be a strongly continuous semigroup on E. Then there exist two constants $K \geq 1$ and $\eta \geq 0$ such that*

$$||T(t)|| \leq K e^{\eta t} \tag{4.2}$$

for $t \geq 0$.

As a direct consequence of the Proposition 4.1 we get the following

Corollary 4.1. *Let* $\{T(t)\}$ *be a strongly continuous semigroup on* E. *Then, for each* $f \in E$, $t \mapsto T(t)f$ *is a continuous function from* $[0, \infty)$ *into* E.

A (possibly unbounded) linear operator $\mathcal{A}$ on E is linear mapping whose domain $\mathcal{D}(\mathcal{A})$ is a subspace of E and whose range $\mathcal{R}(\mathcal{A})$ lies in E. The graph of $\mathcal{A}$ is given by

$$\mathcal{G}(\mathcal{A}) = \{(f, \mathcal{A}f) : f \in \mathcal{D}(\mathcal{A})\} \subset E \times E. \tag{4.3}$$

Note that $E \times E$ is itself a Banach space with componentwise addition and scalar multiplication and norm given by $||(f, g)|| = ||f|| + ||g||$.

Definition 4.4. $\mathcal{A}$ *is said to be* ***closed*** *if* $\mathcal{G}(\mathcal{A})$ *is a closed subspace of* $E \times E$.

When we deal with semigroup theory, a very important quantity is represented by the infinitesimal generator of a semigroup $\{T(t)\}$. Roughly speaking, when $T(t)$ acts on a function f, the infinitesimal generator is reponsible for the evolution along t of the quantity $T(t)f$.

Definition 4.5. *Given a semigroup* $\{T(t)\}$, *the* ***infinitesimal generator*** *of the semigroup* $\{T(t)\}$ *on* E *is the linear operator* $\mathcal{A}$ *defined by*

$$\mathcal{A}f = \lim_{t \to 0} \frac{1}{t} \{T(t)f - f\}. \tag{4.4}$$

The domain $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is the subspace of all $f \in E$ for which the limit (4.4) does exist.

Looking at the infinitesimal generator the link between the semigroup theory and the Markov processes is very clear. In particular, it is possible to give a relation between a suitable operator and a Markov process. Hence, the infinitesimal generator determines, in some ways, the evolution of the observable on that process. On the other hand, when we deal with the valuation equation (3.1), the integro-differential operator $\mathcal{A}$ is the infinitesimal generator of a suitable semigroup $T(t)$.

On the other hand, in the case of Markov processes, when the process is *time-homogeneous*, in force of the so-called *Chapman-Kolmogorov property*, it is possible to find very strong link between making use of the transition function.
The reason why we are focusing on Markov processes is that almost all the models proposed in the modern financial mathematics exhibit the Markov property, as we have seen

in Chapter 2. This reflect the fact that all the informations needed for the evolutions of the process are reflected in its actual values. Furthermore, they are well-understood and a huge literature is available on this topic (see e.g. [35] and the references therein).

Definition 4.6. *Let X_t be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in E, and let $\mathcal{F}_t^X = \sigma(X(s) : s \leq t)$. Then X is a* ***Markov process*** *if*

$$\mathbb{P}\left(X_{t+s} \in \Gamma | \mathcal{F}_t^X\right) = \mathbb{P}\left(X_{t+s} \in \Gamma | X_t\right) \tag{4.5}$$

for all $s, t \geq 0$ and $\Gamma \in \mathcal{B}(E)$.

We notice that the relation (4.5) obviously implies

$$\mathbb{E}\left[f(X_{t+s}) | \mathcal{F}_t^X\right] = \mathbb{E}\left[f(X_{t+s}) | X_t\right] \tag{4.6}$$

for $f \in \mathcal{B}(E)$. Roughly speaking, the Markov property means that the probability density function of the process does not depend on all the previous values reached by the process, but it depends only on the value at the time of the observation.

We notice that the previous properties can be strongened. Considering any stopping time τ at the place of a deterministic time t.

Definition 4.7. *Let X_t, $t \geq 0$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, be an E-valued Markov process with respect to a filtration $\{\mathcal{G}_t\}$ such that X is $\{\mathcal{G}_t\}$-progressive. Suppose $\mathbb{P}(t, x, \Gamma)$ is a transition function for X, and let τ be a $\{\mathcal{G}_t\}$-stopping time with $\tau < \infty$ almost surely. Then X is a* ***strong Markov process*** *at τ if*

$$\mathbb{P}\left(X_{\tau+t} \in \Gamma | \mathcal{G}_\tau^X\right) = \mathbb{P}(t, X_\tau, \Gamma) \tag{4.7}$$

for all $t \geq 0$ and $\Gamma \in \mathcal{B}(E)$, or equivalently

$$\mathbb{E}\left[f(X_{\tau+t}) | \mathcal{G}_\tau^X\right] = \int f(y) \mathbb{P}(t, X_\tau, dy). \tag{4.8}$$

The link between the semigroup theory and the stochastic Markov processes is then given by the following

Definition 4.8. *Let $\{T(t)\}$ be a semigroup on a closed subspace $L \subset \mathcal{B}(E)$. An E-valued Markov process X* ***corresponds*** *to $\{T(t)\}$ if*

$$\mathbb{E}\left[f(X_{t+s}) | \mathcal{F}_t^X\right] = T(s) f(X_t) \tag{4.9}$$

for all $s, t \geq 0$ and $f \in L$.

Then if, for any function $f \in L$, the expected value of $f(X_{t+s})$ can be written as the result of an operator $T(t)$ acting on f, once the natural filtration is known. Hence the stochastic process and the semigroup family given by $T(t)$ are related one to the other.

Remark 4.1. *If $\{T(t)\}$ is given by a transition function, then the relation (4.9), is equivalent to (4.6).*

A very special class of semigroup is given by the Feller semigroup.

Definition 4.9. *Let E be a locally compact and separable metric space. A semigroup of linear positive conservative contraction operator $T(t)$ is a* ***Feller semigroup*** *if, for every $f \in \mathcal{C}_0(E)$ and $x \in E$ it is hold true*

$$T(t)f \in \mathcal{C}_0(E) \tag{4.10}$$

$$\lim_{t \to 0} T(t)f(x) = f(x) \tag{4.11}$$

Hence there exists a particular class of stochastic processes that the one generated by a Feller semigroup.

Definition 4.10 (Feller process)**.** *A Feller process is a Markov process with a transition function associated to a Feller semigroup.*

Several results are shown in [28] dealing with Feller processes and the well-posedness of the martingale problem. In particular we give the following

Definition 4.11 (Martingale solution)**.** *Let X be a measurable stochastic process with values on a given E, defined on a given probability space $(\Omega, \mathcal{F}_t^X, \mathbb{P})$, then we say that X is a solution of the martingale problem $(\mathcal{A}, P_0)$ for a given generator $\mathcal{A}$, and a given initial distribution P_0 if there exists a filtration $\{\mathcal{F}_t\}$ such that X_t is $\{\mathcal{F}_t\}$-adapted and, for each $f \in \mathcal{D}(\mathcal{A})$*

$$f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)ds \tag{4.12}$$

is an $\mathcal{F}_t^X$-martingale.

On the other hand the definition of a stopped martingale problem can be given as well. In particular, consider $\mathcal{A}$ and $S \subset D$ and let X be a stochastic process with initial distribution P_0 in D and sample paths in $\mathcal{D}_D[0, \infty)$. Let τ be the stopping time defined as

$$\tau = \inf\{t \geq 0 : X_t \notin S \text{ or } X_{t^-} \notin S\}. \tag{4.13}$$

Then we can give the following

Definition 4.12 (Stopped martingale solution)**.** *The process X is the solution to the stopped martingale problem for $(\mathcal{A}, P_0, S)$ if*

$$f(X_{t\wedge\tau}) - f(X_0) - \int_0^{t\wedge\tau} \mathcal{A}f(X_{s\wedge\tau})ds \tag{4.14}$$

is an $\mathcal{F}_t^X$-martingale.

Theorem 4.1 (Theorem 6.1, Chapter 4, [28])**.** *Let (E, r) be complete and separable, and let $\overline{\mathcal{A}} \subset \hat{\mathcal{C}}(E) \times \mathcal{B}(E)$. If the $\mathcal{D}_E[0, \infty)$ martingale problem for $\mathcal{A}$ is well-posed, then, for each $\nu \in \mathbb{P}(E)$ and open $U \subset E$ there exists a unique solution of the stopped martingale problem for $(\mathcal{A}, \nu, U)$.*

Theorem 4.2 (Theorem 6.3, Chapter 4, [28])**.** *Let (E, r) be complete and separable, and let $\overline{\mathcal{A}} \subset \hat{\mathcal{C}}(E) \times B(E)$. Let $U_1 \subset U_2 \subset \ldots$ be open subsets of E. Fix $\nu \in \mathbb{P}(E)$, and suppose that for each $k = 1, 2, \ldots$ there exists a unique solution X_k of the stopped martingale problem for $(\mathcal{A}, \nu, U_k)$ with sample paths in $\mathcal{D}_E[0, \infty)$. Setting*

$$\tau_k = \inf\{t \geq 0 : X^k \notin U_k, \text{ or } X^k \notin U_k\},$$

suppose that for each $t > 0$ it is hold true

$$\lim_{k\to\infty} \mathbb{P}\{\tau_k \leq t\} = 0,$$

then there exists a unique solution of the $\mathcal{D}_E[0, \infty)$ martingale problem for $(\mathcal{A}, \nu)$.

4.2 The financial problem and main assumptions

We have seen that many works have been presented that improves our knowledge about the existence of the solution for valuation equation when the classical theory of Sobolev spaces cannot be applied. Unfortunately, all the techniques we have seen in the previous chapter can be applied just on special cases and several cases of interest in finance cannot be dealt with the present literature.

On the other hand, in [17] important improvements of the previous results are given, both from the probabilistic side of the problem (2.9) and from the analytical point of view of the problem (3.1). Due to its importance for our work, and for the mathematical finance in general, we see in details such results in this and the next chapter, and use some of these results in order to improve our knowledge about the regularity of the solutions.

The framework considered in [17] is very general and includes many cases of interest in finance. Roughly speaking, the process X_t is allowed to have sudden jumps, with locally

Lipschitz continuous drift and diffusion coefficients, provided that a Lyapunov type function is put in place. Under such conditions the well-posdness of the martingale problem $(\mathcal{A}, P_0)$ for any initial distribution P_0 is provided, where $\mathcal{A}$ is the infinitesimal generator of the problem (3.1). In particular for any P_0 initial distribution there exists a unique strong Markov process X_t that verifies the martingale problem $(\mathcal{A}, P_0)$. Furthermore it has been proven that the process X_t does not reach the boundary of the domain in a finite time almost surely. Such a result has important implications for the behaviour of X_t and expecially for the evaluation equation (3.1). Indeed, if such hypoteses are verified, then no boundary conditions are needed. Under such hypoteses the existence of a unique viscosity solution is proven. We see such results in the next chapter.

Starting from such results, it is possible to give some estimations on the dependence with respect to the initial data for the proces X_t. Moreover, the presence of the Lyapunov function gives an estimations on the probability that the process X_t approaches the boundary of the domain.

The results provided in [17] can be applied in several pricing problems. The most remarkable examples are represented by

- Path dependent volatility models.
- Jump-diffusion stochastic volatility models.
- Asian option pricing, even in stochastic volatility models.

As stated by the authors, if not for special cases, the actual literature is not able to deal with the previous kind of problem in this generality.

For the rest of the work D is a starshaped open subset of $\mathbb{R}^d$, and is allowed to be unbounded. When the operator $\mathcal{A}$ in (3.2) is a pure differential operator, that is the case when the measure $m(x, dz) = \delta(x)dz$, then the following Assumption on the coefficients will be made:

Assumption 4.1. *Consider $\sigma : D \to \mathbb{R}^{d\times d}$ and $\mu : D \to \mathbb{R}^d$ Lipschitz continuous functions on compact subsets of D. The diffusion matrix $a(x)$ is given by $a(x) = \sigma(x)\sigma(x)^\top$.*

When the operator (3.2) is a full integro-differential operator, we will make in addition the following

Assumption 4.2. *The diffusion matrix $a(x) = (a_{ij}(x))_{i,j=1,\dots,d}$ with $a_{ij} \in \mathcal{C}^2(D)$ and $m : D \to \mathcal{M}(D)$ is continuous and for each h continuous compact supported function in D*

$$\sup_{x\in D} \left| \int_D h(z)m(x, dz) \right| < \infty. \tag{4.15}$$

The hypothesis $a_{ij} \in \mathcal{C}^2(D)$ for all $i, j = 1, \dots, d$ implies Assumption 4.1 (see for instance Proposition 8.25 in [9]). The additional Assumption 4.2 for the regularity of the diffusion matrix is required in the case the process X_t is allowed to have sudden jumps.

As we have already remarked the presence of a Lyapunov-type function $V(x)$ have to be put in place. Applications of Lyapunov functions to stochastic processes can be found in [38]. In our case, such a function have to verify the following structural

Assumption 4.3. *There exists a nonnegative function $V \in \mathcal{C}^2(D)$, such that, for all $x \in D$, the following conditions are hold true*

i) $\int_D V(z)m(x, dz) < \infty$

ii) $\mathcal{A}V(x) \leq C(1 + V(x))$

iii) $\lim_{x \to x_0} V(x) = +\infty$ *for each* $x_0 \in \partial D$

iv) $\lim_{|x| \to \infty} V(x) = +\infty.$

All the previous Assumptions are needed in order to guarantee that the martingale problem $(\mathcal{A}, P_0)$ is well-posed, that is there exists one and only one solution for each initial probability distribution P_0.
As stated in [17], it should be clear that, in most cases, a suitable Lyapunov function verifying the Assumption 4.3 can be built as the sum of a term that controls the process X_t near the boundary, and a term that controls X_t at infinity, but in the general case, these two terms cannot be considered separately. Indeed it may be possible that some components of X_t approach the boundary while some other blow up.

We anticipate that the existence of such a Lyapunov function, verifying the Assumption 4.3, ensures that the stochastic process X_t corresponding to the infinitesimal generator $\mathcal{A}$ in the equation (3.2) does not blow up in a finite time and does not reach the boundary of D. This is the case of several models in finance.
Furthermore, it will be clear that more the Lyapunov function $V(x)$ blows up fastly, more the probability that X_t approaches the boundary vanish rapidly.

The most popular examples in financial literature where the process does not reach the boundary in a finite time are probably the Black and Scholes model and the CIR model. In particular in the CIR model, the value $x = 0$ represents a barrier for the process X_t if some hypotheses on the coefficients are fulfilled. In this model, indeed, if the coefficients κ, θ and σ_0 verify the well-known Feller condition, that is $\sigma_0^2 \leq 2\kappa\theta$, then the stochastic process is constrained to be positive, and X_t cannot be equal to zero in a finite time. In this particular example such a behaviour can be intuitively explained considering that if the volatility σ_0 is low, and the long term level and the mean reverting force are high enough, then, when the process approaches zero, the stochastic fluctuations become small and the mean-reverting force pull the process toward the long term value. We see in Chapter 6 that such a condition ensures the existence of a Lyapunov type condition.

In the general framework we are considering in our assumptions, the presence of such a Lyapunov condition is sufficient in order to guarantee an analogous result.

We notice that, if Assumption 4.3 is satisfied then the condition *(ii)* avoids the process to blow up in a finite time, and forces the death time to be infinity almost surely. This means that no growth condition on the coefficients of the operator $\mathcal{A}$ in (3.2) is needed.

Example 4.1 (Remark 2.4 [17])**.** *An explicit expression of $V(x)$ can be found if the coefficients μ, σ and the measure m verifies some hypotheses. In particular if μ, σ and the jump rate $m(x, D)$ have sublinear growth and the second moment of the jumps is uniformly bounded, that is*

$$m(x, D)^{-1} \int_D |z - x|^2 m(x, dz) < \infty$$

for each x in D, then the process X_t can be controlled at infinity by the function $x \mapsto |x|^2$. Such kind of function obviously verifies all the conditions in Assumption 4.3, except the limit (iii)*. As we have already said, a suitable function have to be added so that the resulting Lyapunov function $V(x)$ is the sum of a barrier function, controlling the behavior near the boundary and a function, controlling the behavior at infinity.*

This is a very specific case. In general, such barrier functions cannot be found separately. As we have already stated, indeed, it may be possible that some component of the process X_t approach the boundary while some other blow up. Such a case is typical for stochastic volatility models.

All the previous remarks are referred to the state space allowed to the stochastic process X_t. However, the presence of the Lyapunov function $V(x)$ has critical implications also for the existence and uniqueness of the viscosity solution to the evaluation equation (3.1). We see such results in Chapters 5 and 6. Furthermore, it determines the growth rate that we can allow for the data (3.2). However, we have to remark that, when the coefficients of the equation (2.9) have a high growth rate, the uniqueness of the solution to the valuation equation (3.1) may be lost. What may happen in such a condition is that the expected value of the discounted final payoff ϕ may be not finite. An example is given in [62] and [53] for the case of stochastic volatility models. On the other hand, Ekström and Tysk in their work [29] give two very important theorems that ensure the uniqueness of a classical solution in the case of stochastic volatility models, when some suitable assumptions on the growth rate of the final payoff is satisfied. We anticipate that in Chapter 6 of our work, we try to generalize their results, and, in the case the process X_t is allowed to have sudden jumps, some results on the viscosity solution may given, if we are able to provide a suitable family of Lyapunov funtions verifying Assumption 4.3. In particular we observe that a family of a suitable Lyapunov function satisfying all Assumptions 4.1-4.3 can be provided whenever the hypoteses of their theorems holds.

The next hypotesis is also assumed to be hold true in [17]. Such assumption is needed in order to guarantee the existence and uniqueness of the viscosity solution to the valuation equation, but it has no implication for the stochastic process X_t. Under the next Assumption the growth rate allowed for the final payoff ϕ and the source term f is bounded by a family of specific functions. In particular an inequality condition with the Lyapunov function $V(x)$ is required. Such a feature is set by the following

Assumption 4.4. *The function f is continuous in $D \times (0,T)$, and c, ϕ are continuous in D and c is bounded from below. Furthermore there exists a strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that*

i) $z \mapsto z\varphi(z)$ *is convex,*

ii) $\lim_{z\to\infty} \varphi(z) = \infty$

iii) $(z_1 + z_2)\phi(z_1 + z_2) \leq C(z_1\varphi(z_1) + z_2\varphi(z_2))$ *for each* $z_1, z_2 \geq 0$.

As a consequence of i) and iii) we get $\varphi(Tz) \leq C_T\varphi(z)$ for each $z \geq 0$. Furthermore, for all $(x,t) \in (D \times (0,T))$ the following conditions are hold true

$$|f(x,t)|\varphi(|f(x,t)|) \leq C_T(1 + V(x)) \tag{4.16}$$

$$|\phi(x)|\varphi(|\phi(x)|) \leq C(1 + V(x)) \tag{4.17}$$

As it is clear, if the function $V(x)$ has a high growth rate, then the final payoff is allowed to blow up fastly. On the other hand, due to representation formula, this means that the process X_t has finite higher moments.

Remark 4.2 (Remark 2.6 [17])**.** *We observe that the functions $\varphi(z) = z^\alpha$ with $\alpha > 0$ and $\varphi(z) = \log(z + \alpha)$ with $\alpha > 1$ are suitable functions that may verify the Assumption 4.4.*

Example 4.2. *If the coefficients $\mu(x)$, $m(x,D)$ and $a(x)$ verify the sublinear/subquadratic growth assumptions previously discussed, then ϕ and f are allowed to have polynomial growth rate up to order $q < 2$. This is obviously achieved since in such condition the term of the Lyapunov function that controls the growth rate at infinity is given by $|x|^2$. Then if we take, $\varphi(|\phi(x)|) = |\phi(x)|^\alpha$, with $\alpha > 0$, we have*

$$|\phi(x)|^{q+\alpha} \leq C_T \left(1 + |x|^2\right)$$

that implies $q < 2$, and the same for $f(x,t)$.

In the rest of the thesis, we assume all the hypotheses Assumption 4.1-4.4 be hold true, unless differently specified.

The next result is devoted to give sufficient conditions that guarantee the well-posedness of the martingale problem $(\mathcal{A}, P_0)$ for each initial distribution P_0, that is there exists a unique solution to the considered martingale problem. Furthermore, the solution X_t is a strong Markov process, and if the initial distribution $P_0 = \delta(x)$ of the process X_t is included in the interior of the domain D, then the solution does not reach the boundary or blow up in a finite time.

In what follows, it may be useful consider not just the original martingal problem, but a localized problem. In particular, it is possible to consider a localized operator $\mathcal{A}^{oc}$ that determines the evolution of a process $\tilde{X}_t$.

For every $z \in \mathbb{R}_+$, we define the domain D_z as

$$D_z = \{x \in D : V(x) < z\} \tag{4.18}$$

where $V(x)$ is the Lyapunov function that verifies Assumptions 4.1-4.4. Since the value of z can be arbitrarly chosen, then we can suppose without loss of generality that z is large enough that the domain D_z is nonempty. Due to the definition of D_z and D, the domain D_z is an open subset with closure included in D.

Now we consider the non-negative compact supported smooth function $\xi_z(x) : \mathbb{R}^d \to \mathbb{R}$, defined as

$$\xi_z(x) = \begin{cases} 1 & x \in D_z \\ \leq 1 & x \notin (D_z)^c \cap D_{z+1} \\ 0 & x \notin D_{z+1}. \end{cases}$$

Then, we consider the new coefficients $\tilde{\mu}(x)$, $\tilde{\sigma}(x)$ and the measure $\tilde{m}(x, dz)$ given by

$$\tilde{\mu}(x) = \xi_z(x)\mu(x) \tag{4.19}$$

$$\tilde{\sigma}(x) = \sqrt{\xi_z(x)}\sigma(x) \tag{4.20}$$

$$\tilde{m}(x, dw) = \xi_z(x)m(x, dw). \tag{4.21}$$

Remark 4.3. *The localized coefficients $\tilde{\mu}(x)$, $\tilde{\sigma}(x)$ have sublinear growth rate. Furthermore $\tilde{\mu}(x)$, $\tilde{\sigma}(x)$ and $\tilde{m}(x, dw)$ coincide with $\mu(x)$, $\sigma(x)$ and $m(x, dw)$ for each x in D_z.*

Now we consider the localized martingale problem, where the process $\tilde{X}$ is solution of the SDE (2.9) and is driven by the localized coefficients $\tilde{\mu}(x)$, $\tilde{\sigma}(x)$ and $\tilde{m}(x, dw)$. Then the infinitesimal generator $\tilde{\mathcal{A}}^{loc}$ is given by

$$\tilde{\mathcal{A}}^{loc} = \tilde{\mathcal{A}}_d^{loc} + \tilde{\mathcal{J}}^{loc} \tag{4.22}$$

where the localized operators $\tilde{\mathcal{A}}_d^{loc} : \mathcal{C}_c^2(\mathbb{R}^d) \to \mathcal{C}_c(\mathbb{R}^d)$ and $\tilde{\mathcal{J}}^{loc} : \mathcal{C}_c^2(\mathbb{R}^d) \to \mathcal{C}_c(\mathbb{R}^d)$ are defined, for each $g(x) \in \mathcal{C}_c^2(\mathbb{R}^d)$ as

$$\begin{aligned}
\tilde{\mathcal{A}}_{d,z}^{loc} g(x) &= \xi_z(x) \left[\langle \nabla g(x), \mu(x) \rangle + \frac{1}{2} tr \left(\nabla^2 g(x) a(x) \right) \right] \\
\tilde{\mathcal{J}}_z^{loc} g(x) &= \xi_z(x) \int_D \left[g(z) - g(x) \right] m(x, dz).
\end{aligned}$$

Roughly speaking, the localized operators $\tilde{\mathcal{A}}_{d,z}^{loc}$ and $\tilde{\mathcal{J}}_z^{loc}$ are respectively the diffusion part and the integral part of the operator $\mathcal{A}$ defined in (3.2) restricted to any domain D_z defined in (4.18).

Considering a localized operator is a technique that is used in general to get some localized properties of the stochastic process or several quantities related to it. In particular such a technique is followed in [17] in order to prove the well-posedness of the martingale problem, or in Baldi [9] where the technique is used in order to prove some properties of continuity of the process X. This procedure will be used also in order to get our main results of regularity in this chapter, and in the next one.

Theorem 4.3 (Theorem 2.8 [17]). *Suppose Assumptions 4.1-4.4 be hold true. Let $\mathcal{A}$ be the operator defined by (3.2). Then, for every probability distribution P_0 on D, there exists one, and only one stochastic process X, solution of the martingale problem for $(\mathcal{A}, P_0)$ with $\mathcal{D}(\mathcal{A}) = \mathcal{C}_c^2(D)$.*
Furthermore X is a homogeneous strong Markov process with paths in $\mathcal{D}_D[0, \infty)$.
Denoting by X^x the process with $P_0 = \delta(x)$, where $x \in D$, it holds, for every $T \geq 0$, and $\{\mathcal{F}_t^{X^x}\}$-stopping time τ

$$\sup_{0 \leq t \leq T} \mathbb{E} \left[V(X_{t \wedge \tau}^x) \right] \leq C_T (1 + V(x)). \tag{4.23}$$

Given the definition of $V(x)$, the inequality (4.23) means that, if the starting point of X^x is in the interior of the domain D, then the expected value of $\mathbb{E} \left[V(X_{t \wedge \tau}^x) \right]$ is bounded for each $t \wedge \tau$, with any $\{\mathcal{F}_t^{X^x}\}$-stopping time τ. Furthermore, since the Lyapunov function $V(x)$ blows up at the boundary, and at infinity, the inequality (4.23) yields that the probability that the stochastic process X^x reaches the boundary or blows up in a finite time is $\mathbb{P}$-negleactable.

We want to give some comments on Theorem 4.3, pointing our attention to the main difficulties the authors have overcome, due to the fact that only Assumptions 4.1-4.4 are assumed to be satisfied. Furthemore, for the rest of our work, we need to highlight the main results that the authors have given in the proof of the theorem.

We observe first of all that no assumptions are made on the rank of the diffusion matrix $a(x)$ that is allowed to be degenerate. Furthermore the coefficients are allowed to loose their Lipschtiz continuity near the boundary and to be unbonded with fast growing rate at infinity. As a consequence, as stated in [17], standard theorems on Lévy generators fail.

From an intuitive point of view, the presence of a positive-defined diffusion matrix and Lipschitz continuous coefficients, should have a regularizing effect to the final distribution of the process X_t. Unfortunatelly this is not the case. On the other hand, such a regularizing effect is definitely lost, in general, when sudden jumps are considered in the evolution of the process. Indeed, it is shown in [17] that theorems on Lévy generators imply the well-posedness only for the localized martingale problem $(\tilde{\mathcal{A}}_z^{loc}, P_0, D_z)$, for any initial distribution P_0, whenever the coefficients of the generator (3.2) are locally Lipschitz continuous, and the jumps are not allowed. However, at the presence of the integral term, stronger assumptions have to be made, and the authors are forced to require the diffusion matrix be twice differentiable.

The strategy of the proof given in [17] is to show, first of all, the well-posedness of the martingale problem for any localized problem $(\tilde{\mathcal{A}}_z^{loc}, P_0, D_z)$ for any initial distribution P_0 on D. This is done making use of Lévy generators theorems in order to prove that Assumptions 4.1-4.3 ensure that the localized operator $\tilde{\mathcal{A}}_z^{loc}$ generates a Feller semigroup and the solution X_t is a Feller process. In doing this, the fact that the domain D is starshaped plays a crucial rule. We observe that this assumption does not affect the generality of the results, at least for almost all cases of interest in finance.

The goal of the second step of the proof given in [17] is mainly to show that the well-posedness of the localized martingale problem can be extended to the original problem defined in D.

In this part of the proof, the existence of the Lyapunov function $V(x)$ verifying Assumptions 4.1-4.4 is required. In particular, for each $x \in D$, we indicate by X_t^x the solution of the stopped martingale problem for $(\tilde{\mathcal{A}}_z^{loc}, \delta(x), D_z)$. Then, given a sequence of increasing domains D_z defined as in (4.18), the corresponding sequence of stopping time τ_z^x defined as

$$\tau_z^x = \inf\{t \geq 0 : X_t^x \notin D_z, \text{ or } X_{t^-}^x \notin D_z\}, \tag{4.24}$$

represents the first exit time of the process X_t^x from the domains D_z.

Then, making use of Gronwall's lemma, under Assumption 4.3, in [17] it is shown the following relation

$$\mathbb{E}\left[V(X^x_{T\wedge\tau^x_z}) + V(X^x_{(T\wedge\tau^x_z)^-})\right] \le 2\,(V(x) + C\cdot T)\,e^{C\cdot T}. \tag{4.25}$$

We observe that the term $V(X^x_{(T\wedge\tau^x_z)^-})$ is introduced by the nonlocal term of the operator. In this case some additional algebra is needed in order to properly take into account the domain D_z. As a consequence of the equation (4.25) it is possible to give an estimate on the probability that the process X_t reaches the boundary in a finite time. In particular the following inequality holds

$$\mathbb{P}\{\tau^x_z \le T\} \le \frac{2\,(V(x) + C\cdot T)\,e^{C\cdot T}}{z}. \tag{4.26}$$

The equation (4.26) has two main consequences. The first one is immediate that is letting z to infinity $\mathbb{P}\{\tau^x_z \le T\} \to 0$. Furthermore, we observe that when z goes to infinity, the domain D_z converges to D. In other words, the probability that the stochastic process X^x reaches the boundary of the domain D or blows up in a finite time is $\mathbb{P}$-negleactable.
The second consequence is a direct application of Theorem 4.2, that is the martingale problem for $(\mathcal{A}, \delta(x))$ has one and only one solution in $\mathcal{D}_D[0,\infty)$.

Hence Assumptions 4.1-4.3 ensure that the martingale problem $(\mathcal{A}, P_0)$ is well-posed and then the existence of a weak solution to the SDE (2.9) is granted.

4.3 Continuous dependence results for the pure diffusive case

In the previous section we have seen that there exists a unique solution to the martingale problem $(\mathcal{A}, P_0)$ for every initial distribution P_0. Furthermore, such a solution is a strong Markov process, with paths in $\mathcal{D}_D[0,\infty)$ almost surely.

Unfortunately, such a result is not strong enough in order to guarantee that there exists a strong solution to the stochastic differential equation (2.9). Indeed Theorem 4.3 ensures only the existence of a weak solution. This means that for every initial distribution there exists a probability space $(\Omega, \mathcal{F}^X_t, \mathbb{P})$ on which the solution is defined, but such a probability space is not *a priori* defined.

We indicate by X^x_s the stochastic process at time s, solution of the problem (2.9), starting at x. Moreover in the rest of the present chapter, we will focus on the pure diffusive case $\gamma \equiv 0$.

Our objective is to establish some estimates concerning with the dependence of the solution X^x on the initial datum x. Under the same assumptions given in [17], we are able to state a Lipschitz-type regularity of the process $\{X_s^x\}_s$ with respect to x.

For each $z > 0$, consider a countable set S such that $S \subset D_z$ where D_z is the domain defined in (4.18). Clearly $S \cap \partial D = \{\emptyset\}$. Then we introduce the following notations:

$$\tau_z^S = \inf\left\{ s \geq 0 \ : \ \sup_{x\in S} V(X_s^x) \geq z \right\}, \tag{4.27}$$

$$\Delta X(s, x_1, x_2) = |X_s^{x_1} - X_s^{x_2}|, \qquad \forall\, x_1,\ x_2 \in S,\ \ s \geq 0. \tag{4.28}$$

When S is the singleton $\{x\}$, we shall use the notation τ_z^x. Clearly τ_z^x concides with the stopping time defined in (4.24), in the case no sudden jumps are considered.

Theorem 4.4. *Suppose that Assumptions 4.1-4.3 are satisfied. Suppose furthermore that there exists a strong solution to the SDE (2.9) with $\gamma \equiv 0$. Then it hold:*

i) For every $z > 0$, $T \geq 0$, $p \geq 1$ and for every countable set $S \subset D_z$,

$$\sup_{0 \leq t \leq T} \mathbb{E}\left[\left(\Delta X(t \wedge \tau_z^S, x_1, x_2)\right)^p \right] \leq e^{\kappa_z T} |x_1 - x_2|^p, \tag{4.29}$$

for any $x_1, x_2 \in S$. Here the constant $\kappa_z > 0$ depends only on p and the Lipschitz constants of μ and σ on the domain D_z.

ii) If there exists $p^* > 0$ and a constant $\tilde{C}$ such that $|x|^{p^*} \leq \tilde{C}\,(1 + V(x))$. Then, for each compact subset $K \subset D$ and for every $T > 0$,

$$\lim_{h\to 0^+} \sup_{\substack{x_1, x_2 \in K, \\ |x_1 - x_2| \leq h}} \mathbb{E}\left[\left(\Delta X(T, x_1, x_2)\right)^{p^*} \right] = 0. \tag{4.30}$$

iii) If the coefficients $\mu(x)$ and $\sigma(x)$ have at most linear growth rate with respect to x, then for every $T \geq 0$, $p \geq 1$ and for each $K \subset D$ compact,

$$\sup_{\substack{x \in K \\ 0 \leq t \leq T}} \mathbb{E}\left[|X_t^x|^p\right] \leq C_{T,p}(1 + |x|^p). \tag{4.31}$$

where the constant $C_{T,p}$ depends only on T and p.

iv) For every $z > 0$, $T \geq 0$, $p \geq 2$ and for every $x \in D_z$, it holds

$$\mathbb{E}\left[\left|X^x_{t\wedge\tau^x_z} - X^x_{s\wedge\tau^x_z}\right|^p\right] \leq L_{z,p,T}|t-s|^{p/2}, \tag{4.32}$$

for any $0 \leq s \geq t \leq T$. Here the positive constant $L_{z,p,T}$ depends only on the maximum of μ and σ on the domain D_z, p and T.

The first two points in the Theorem 4.4 are mainly meaningful. Such results yield that for each countable subset of D, the expected value of the distance between two processes starting from two near points x and y is bounded by a Lipschitz-type condition up to the first exit time from the domain D_z. The limit in (4.30) yields a uniform-type continuity result with respect to the initial datum. The assertion in (iii) is, in fact, a classical result in the theory of stochastic processes, and it is proved, for istance, in Baldi [9] for the pure diffusive case and in Pham [57] for the case of jumps. However, in the the literature of stochastic processes, the results of regularity with respect to the initial datum take into account global Lipschitz continuous coefficients. Weak conditions are available only in the one-dimensional case, see for instance [63].

The direct consequence of Theorem 4.4 concerns with the case of global Lipschitz continuous coefficients.

Corollary 4.2. *Suppose that the hypotheses of Theorem 4.4 are fulfilled. Suppose furthermore that the coefficients $\mu(x)$ and $\sigma(x)$ are Lipschitz continuous on D. Then, for every $p \geq 1$, $T \geq 0$ it holds:*

$$\mathbb{E}\left[\Delta X(T, x_1, x_2)^p\right] \leq |x-y|^p e^{\kappa_* T}. \tag{4.33}$$

where the constant κ_* depends on p and the Lipschitz constants of μ and σ.

An analogous result for jump processes is proved by Pham [57] (see Lemma 3.1).

***Proof of Theorem 4.4** (i)* - The proof of Theorem 4.4 is mainly based on the techniques used by T. Yamada and S. Watanabe in [63] to state the uniqueness of stochastic differential equations in presence of merely uniformly continuous coefficients.

Fix a countable subset S of D_z, $0 \leq s \leq T$ and $x_1,\ x_2 \in S$. Let τ^S_z be the stopping time (4.27). Consider the sequence $\ell_n = 2/(n^2+n+2)$, $n \geq 0$. Clearly the sequence $\{\ell_n\}_{n\geq 0}$ satisfies $1 = \ell_0 > \ell_1 > \ell_2 > \cdots > \ell_n \to 0$, as $n \to +\infty$. We also observe, by a direct calculation, that

$$\int_{\ell_n}^{\ell_{n-1}} \frac{1}{w^2}\, dw = n. \tag{4.34}$$

Then consider a twice continuously differentiable function $g_n : [0,+\infty) \to [0,+\infty)$ such that $g(0) = 0$,

$$g_n'(u) = \begin{cases} 0, & 0 \le u \le \ell_n \\ \text{between 0 and 1}, & \ell_n < u < \ell_{n-1} \\ 1, & u > \ell_{n-1} \end{cases} \tag{4.35}$$

$$g_n''(u) = \begin{cases} 0, & 0 \le u \le \ell_n \\ \text{between 0 and } \frac{1}{n}u^{-2}, & \ell_n < u < \ell_{n-1} \\ 0, & u \ge \ell_{n-1} \end{cases} \tag{4.36}$$

When we consider the extension to $x \in \mathbb{R}^d$, we choose $g_n(|x|^p)$, $p \ge 1$, twice differentiable such that $g_n(|x|^p) \le g_{n+1}(|x|^p) \to |x|^p$, as $n \to +\infty$, for any $x \in \mathbb{R}^d$.

In the following, we denote:

$$\begin{aligned} Y_\lambda &= X_\lambda^{x_1} - X_\lambda^{x_2}, \\ \Delta\mu_\lambda &= \mu\big(X_\lambda^{x_1}\big) - \mu\big(X_\lambda^{x_2}\big), \\ \Delta\sigma_\lambda &= \sigma\big(X_\lambda^{x_1}\big) - \sigma\big(X_\lambda^{x_2}\big), \end{aligned}$$

for $0 \le \lambda \le T$. Applying Itô's formula to $\lambda \mapsto g(|Y_\lambda|^p)$, we have

$$\begin{aligned} g_n(|Y_{\lambda\wedge\tau_z^S}|^p) &= g_n(|x_1 - x_2|^p) + \int_0^{\lambda\wedge\tau_z^S} \Bigg\{ \frac{p^2}{2} g_n''(|Y_s|^p)|Y_s|^{2p-4} \left|\Delta\sigma_s^\top Y_s\right|^2 ds \\ &+ \left[p\, g_n'(|Y_s|^p)|Y_s|^{p-2}\langle\Delta\mu_s, Y_s\rangle\right] \Bigg\} ds \\ &+ \frac{p}{2}\int_0^{\lambda\wedge\tau_z^S} g_n'(|Y_s|^p)|Y_s|^{p-2} \left\{ tr\left(\Delta\sigma_s\Delta\sigma_s^\top\right) + \frac{p-2}{|Y_s|^2}\left|\Delta\sigma_s^\top Y_s\right|^2 \right\} ds \\ &+ \int_0^{\lambda\wedge\tau_z^S} p\, g_n'(|Y_s|^p)|Y_s|^{p-2} Y_s^\top \Delta\sigma_s\, dW_s \\ &= g_n(|x_1 - x_2|^p) + I_1 + I_2 + I_3. \end{aligned} \tag{4.37}$$

where I_1, I_2 and I_3 are the integral terms in (4.37) and are equivalent to

$$I_1 = \frac{p^2}{2}\int_0^{\lambda\wedge\tau_z^S} g_n''(|Y_s|^p)|Y_s|^{2p-4}\left|\Delta\sigma_s^\top Y_s\right|^2 ds + \int_0^{\lambda\wedge\tau_z^S}\left[p\, g_n'(|Y_s|^p)|Y_s|^{p-2}\langle\Delta\mu_s, Y_s\rangle\right] ds \quad (4.38)$$

$$I_2 = \frac{p}{2}\int_0^{\lambda\wedge\tau_z^S} g_n'(|Y_s|^p)|Y_s|^{p-2}\left\{tr\left(\Delta\sigma_s\Delta\sigma_s^\top\right) + \frac{p-2}{|Y_s|^2}\left|\Delta\sigma_s^\top Y_s\right|^2\right\} ds \quad (4.39)$$

$$I_3 = \int_0^{\lambda\wedge\tau_z^S} p\, g_n'(|Y_s|^p)|Y_s|^{p-2}Y_s^\top\Delta\sigma_s\, dW_s. \quad (4.40)$$

We remark that when $p = d = 1$, then the integrand in I_2 is identically zero that is the case coming from unidimensional Ito's Lemma.
We notice that the functions $\mu(x)$, $\sigma(x)$ are locally Lipschitz continuous with respect to x with constants K_μ and K_σ. Such constants depend on the considered domain D_z. We indicate by $K_z = \max(K_\mu, K_\sigma)$.

If $s \le \lambda\wedge\tau_z^S$, then $s \le \lambda$, which implies $s = s\wedge\tau_z^S$. Hence, we get

$$|I_1| \le 2p\, K_z\int_0^{\lambda\wedge\tau_z^S}|Y_{s\wedge\tau_z^S}|^p ds + \frac{p^2}{2n}K_z^2\lambda, \quad (4.41)$$

$$|I_2| \le \frac{p}{2}K_z^2\,(d+p-2)_+\int_0^{\lambda\wedge\tau_z^S}|Y_{s\wedge\tau_z^S}|^p ds, \quad (4.42)$$

where $(x)_+$ denotes the positive part of x. Taking the expectation of the relation (4.37), by the optimal stopping theorem (see Theorem 4.6. in [64]) we get $\mathbb{E}[I_3] = 0$. By applying the monotone convergence theorem and taking the limit for n towards infinity, for all $0 \le t \le T$ we obtain

$$\mathbb{E}\left[g_n(|Y_{t\wedge\tau_z^S}|^p)\right] \underset{n\to+\infty}{\longrightarrow} f(t) \equiv \mathbb{E}\left[\left(\Delta X(t\wedge\tau_z^S, x_1, x_2)\right)^p\right]. \quad (4.43)$$

Now we take the expectation in (4.37), and letting $n\to+\infty$, by using (4.41) and (4.42) we get the inequality

$$f(t) \le |x_1 - x_2|^p + \kappa_z\int_0^t f(u)du, \quad (4.44)$$

for any $t\in[0,T]$. Here κ_z denotes the constant coefficients $pK_z[2 + \frac{K_z}{2}(d+p-2)_+]$. Finally, the relation (4.29) follows directly by Gronwall's inequality.

(ii) - Let κ_z be the costant defined in the equation (4.31). Let us fix $K\subset D$ compact and $T>0$. The coefficients μ and σ are locally Lipschitz continuous, then for every $n\in\mathbb{N}$

we can always define an increasing sequence $z_n > 0$ such that the following conditions are satisfied:

$$\lim_{n\to\infty} z_n = \infty, \tag{4.45}$$

$$\lim_{n\to\infty} \frac{e^{\kappa z_n T}}{n^{p^*}} = 0. \tag{4.46}$$

Indeed D_{z_n} is an increasing sequence of subsets converging to D, hence there exists $\bar{n}$ such that, for each $n \geq \bar{n}$ we have $K \subset D_{z_n}$.

Suppose for contradiction that (4.30) does not hold. Therefore, we can find $\varepsilon > 0$ such that for every $n \in \mathbb{N}$, there exist $x_{1,n},\ x_{2,n} \in K$ satisfying,

$$|x_{1,n} - x_{2,n}| \ \leq \ \frac{1}{n}, \qquad \text{and} \qquad \mathbb{E}\left[(\Delta X(T, x_{1,n}, x_{2,n}))^{p^*} \right] > \varepsilon. \tag{4.47}$$

Let us consider the set $S_n = \{x_{1,n},\ x_{2,n}\}$ with the stopping time $\tau_{z_n}^{S_n}$ as defined in (4.27). Since $\{\tau_{z_n}^{S_n} < T\} \subset \{\tau_{z_n}^{x_{1,n}} \leq T\} \cup \{\tau_{z_n}^{x_{2,n}} \leq T\}$, for every $\lambda > 0$, the inequalities (4.26) and (4.29) yield

$$\begin{aligned}
\mathbb{P}\left(\Delta X(T, x_{1,n}, x_{2,n}) > \lambda\right) \ &\leq \ \mathbb{P}\left(\Delta X(T, x_{1,n}, x_{2,n}) > \lambda,\ \tau_{z_n}^{S_n} \geq t\right) + \mathbb{P}\left(\tau_{z_n}^{S_n} < T\right) \\
&\leq \ \frac{1}{\lambda^{p^*}}\ \mathbb{E}\left[(\Delta X(T \wedge \tau_{z_n}^{S_n}, x_{1,n}, x_{2,n}))^{p^*} \right] \\
&+ \ \mathbb{P}\left(\tau_{z_n}^{x_{1,n}} \leq T\right) + \mathbb{P}\left(\tau_{z_n}^{x_{2,n}} \leq T\right)
\end{aligned} \tag{4.48}$$

and then

$$\mathbb{P}\left(\Delta X(T, x_{1,n}, x_{2,n}) > \lambda\right) \ \leq \frac{e^{\kappa^*_{z_n} T}}{\lambda^{p^*}\, n^{p^*}} + \frac{1}{z_n}\left[V(x_{1,n}) + V(x_{2,n}) + 2CT\right] e^{CT} \tag{4.49}$$

where C is the costant given in Assumption 4.3 which is, in particular, independent of n. Thus, by letting n go to infinity, in force of the relations (4.45)-(4.46) and still using Assumption 4.3, we have

$$\lim_{n\to+\infty} \mathbb{P}\left(\Delta X(T, x_{1,n}, x_{2,n}) > \lambda\right) = 0. \tag{4.50}$$

Therefore the sequence of random variables $\{(\Delta X(T, x_{1,n}, x_{2,n}))^{p^*}\}_n$ converges to zero in probability and, by the inequality $|x|^{p^*} \leq \tilde{C}(1 + V(x))$, it is uniformly integrable. In fact, it is bounded by the random variable $\tilde{C}'\left(1 + V(X_t^{x_{1,n}}) + V(X_t^{x_{2,n}})\right)$, with $\tilde{C}'$ independent of n. By the inequality (4.23), the expected value of $V(X_t^{x_{i,n}})$, for $i = 1, 2$, is bounded by $C_T(1 + \max_{x\in K} V(x))$, for any n. Thus, the expected value of $(\Delta X(T, x_{1,n}, x_{2,n}))^{p^*}$ goes to zero, as $n \to \infty$, in contrast with (4.47).

(iii) - This result can be achieved by standard arguments. In particular it is sufficient to apply Itô's lemma and then the result comes from Gronwall's inequality. Such a procedure can be found in Proposition 8.15 in [9] for $p \geq 2$, in [57] for uniformly Lipschitz continuous coefficients and $p \in [0, 2]$.

(iv) - We prove the estimate in the case $p = 2$. The general case can be proved with similar arguments. Let $T \geq 0$. Since the process X_t^x is solution of the SDE (2.9), for every $0 \leq s \leq t \leq T$, we have

$$\begin{aligned} \left| X^x_{t\wedge\tau^x_z} - X^x_{s\wedge\tau^x_z} \right|^2 &\leq \left| \int_{s\wedge\tau^x_z}^{t\wedge\tau^x_z} \mu(X^x_u)du + \int_{s\wedge\tau^x_z}^{t\wedge\tau^x_z} \sigma(X^x_u)dW_u \right|^2 \\ &\leq 2 \sup_{y\in D_z} |\mu(y)|^2 |t-s|^2 + 2 \left| \int_{s\wedge\tau^x_z}^{t\wedge\tau^x_z} \sigma(X^x_u)dW_u \right|^2 . \end{aligned} \tag{4.51}$$

Taking the expected value in (4.51) and applying Itô's isometry for stopping times (see Theorem 4.2 in [35]), we get

$$\begin{aligned} \mathbb{E} \left| X^x_{t\wedge\tau^x_z} - X^x_{s\wedge\tau^x_z} \right|^2 &\leq 2 \sup_{y\in D_z} \left[|\mu(y)|^2 + |\sigma(y)|^2 \right] \max(|t-s|^2, |t-s|) \\ &\leq L_{z,T}|t-s|, \end{aligned} \tag{4.52}$$

where $L_{z,T} = 2\sup_{y\in D_z} \left[|\mu(y)|^2 + |\sigma(y)|^2 \right] \max(2T, 1)$. ■

Remark 4.4. *Applying Hölder inequality, it is possible to get the point (iv) of Theorem 4.4 for every* $p > 0$.

Remark 4.5. *If the assumption on the growth rate of the coefficients* μ *and* σ *of the point iii) are verified, then the point ii) is verified for each* $p^* \geq 1$. *This is true since, in this case, the sequence of variables* $\{(\Delta X(T, x_{1,n}, x_{2,n}))^{p^*}\}_n$ *can be bounded by* $2\sup_{x\in S} |X_t^x|^{p^*}$ *that has finite expectation in force of the the Theorem 4.4-(iii).*

Remark 4.6. *Under the same hypotheses at the point i) of Theorem 4.4, it is possible to prove the following inequality*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \left(\Delta X(t\wedge\tau_z^S, x_1, x_2) \right)^p \right] \leq e^{\kappa_z T} |x_1 - x_2|^p. \tag{4.53}$$

This result is achieved following the same procedure used for the point i) in Theorem 4.4, and by applying a Doob inequality to $\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_z^S} I_3(t) \right]$.

4.4 Weighted continuity estimates for the process X

In the previous section, we have seen that if the coefficients are Lipschitz continuous, and for the diffusive case ($\gamma \equiv 0$), then the distance between two processes starting from two points near one to each other, is in some way Lipschitz continuous (Theorem 4.4 and Corollary 4.2). Such a result is similar to the one obtained for Ordinary Differential Equations, where the local Lipschitz regularity of the coefficiens ensures the existence of a local solution, providing a constraint on dependence on the initial data. We remark that, in the case of Corollary 4.2, the maximum expected distance of the two solutions is independent on the wideness of the considered domain and it depends on the time only.

On the other hand, the result *(i)* of Theorem 4.4 is in some way very close to the Corollary 4.2, but the coefficients are allowed to be only locally Lipschitz continuous. In such a case, the result is weakened of course, and the regularity with respect to the initial data is hold up to a given stopping time. Unfortunately, this result does not guarantee that the regularity estimate is satisfied at any.

In this section we state a result that provides, under the same assumptions of Theorem 4.4-*(i)*, a uniform Lipschitz-type estimate for the expected value of a *weighted* distance between two processes starting at different initial data. The weight process is based on the Lyapunov function $V(x)$.

Theorem 4.5. *Suppose Assumptions 4.1-4.3 be hold true. Suppose furthermore that there exists a strong solution to the problem (2.9), with $\gamma \equiv 0$. For each $z > 0$ let $L_\mu(z)$ and $L_\sigma(z)$ be the Lipschitz constants of $\mu(x)$ and $\sigma(x)$ on D_z. Then, for each compact and countable set $S \subset D$,*

$$L_t^S = p\left[\sup_{x\in S} L_\mu\left(V(X_t^x)\right) + \frac{1}{2} L_\sigma^2\left(V(X_t^x)\right)(d+p-2)_+\right], \tag{4.54}$$

is an $\{\mathcal{F}_t^X\}_t$-adapted process, and for every $p \geq 1$, $x_1, x_2 \in S$, the process X_t satisfies

$$\mathbb{E}\left[e^{-\int_0^t L_\tau^S d\tau}|\Delta X\left(t, x_1, x_2\right)|^p\right] \leq |x_1 - x_2|^p, \qquad \forall\, t \geq 0. \tag{4.55}$$

Remark 4.7. *If the coefficients μ and σ are globally Lipschitz continuous, then the process L_t^S is bounded for every subset S and the previous result reduces to Corollary 4.2.*

Proof of Theorem 4.5. The functions $L_\mu(z),\ L_\sigma(z)$ are clearly nondecreasing functions of $z > 0$ and $V(x)$ is in particular continuous. Therefore the fact that L_t^S is $\{\mathcal{F}_t^X\}_t$-adapted is easily argued.

As in the proof of Theorem 4.4, let us consider the sequence $\ell_n = 2/(n^2 + n + 2)$ which guarantees the relation

$$\int_{\ell_n}^{\ell_{n-1}} z^{-2} = n, \qquad \forall\, n \geq 1. \tag{4.56}$$

Let $\{g_n\}_n$ be the sequence of functions defined as in (4.35) and (4.36), where the upper bound of $g_n''(w)$ is $\frac{1}{n}w^{-2}$, for any $w \geq 0$ and $n \geq 1$.

Let S be a countable compact subset of D. We consider $\bar{z} > 0$ such that $S \subset D_z$, for any $z > \bar{z}$. For every $t \geq 0$, set $\xi_t^z = e^{-\int_0^t L_\tau^S d\tau - z^{-1}t}$, then by the same calculations followed in the proof of Theorem 4.4, for every $0 \leq \lambda \leq t \wedge \tau_z^S$ and for all $x_1,\ x_2 \in S$, we can apply Itô's formula leading to the equation

$$\begin{aligned}
\xi_\lambda^z g_n(|Y_\lambda|^p) &= g_n(|x_1 - x_2|^p) + \int_0^\lambda \Big\{ p\, g_n'(|Y_s|^p)|Y_s|^{p-2}\langle \Delta\mu_s, Y_s\rangle \\
&+ \frac{p}{2} g_n'(|Y_s|^p)|Y_s|^{p-2}\left[tr\left(\Delta\sigma_s \Delta\sigma_s^\top\right) + \frac{p-2}{|Y_s|^2}\left|\Delta\sigma_s^\top Y_s\right|^2 \right] \\
&- L_{s\wedge\tau_z^S} g_n(|Y_s|^p) \Big\}\, \xi_s^z\, ds \\
&+ \frac{p^2}{2}\int_0^\lambda g_n''(|Y_s|^p)|Y_s|^{2p-4}\left|\Delta\sigma_s^\top Y_s\right|^2\ \xi_s^z\, ds \\
&+ \int_0^\lambda p\,\xi_s^z\ g_n'(|Y_s|^p)|Y_s|^{p-2} Y_s^\top \Delta\sigma_s\, dW_s \\
&= g_n(|x_1 - x_2|^p) + I_1 + I_2 + I_3.
\end{aligned} \tag{4.57}$$

As a direct consequence of the definition of the process L_s, for the integral term I_1, we have the estimate

$$|I_1| \leq \int_0^\lambda L_{s\wedge\tau_z^S}\left[|Y_s|^p - g_n(|Y_s|^p)\right]\xi_s^z\, ds. \tag{4.58}$$

For the second integral term in (4.57), we have the estimate

$$|I_2| \leq \frac{p^2}{2n}\int_0^\lambda L_\sigma^2\left(\sup_{x\in S} V(X_{s\wedge\tau_z^S})\right)\xi_s^z\, ds. \tag{4.59}$$

Since for every $z > \bar{z}$, $L_{s\wedge\tau_z^S}\,\xi_s^z$ and $L_\sigma^2\left(\sup_{x\in S} V(X_s \wedge \tau_z^S)\right)\xi_s^z$ are bounded processes and almost surely $g_n(|Y_s|^p)$ converges increasingly to $|Y_s|^p$ for $n \to \infty$, we easily obtain that $\mathbb{E}[|I_i|] \to 0$, as $n \to \infty$, for $i = 1, 2$.

Thus, taking the expectation in (4.57) and then the limit with respect to $n \to \infty$, as in

Theorem 4.4, we get the inequality

$$\mathbb{E}\left[e^{-\int_0^{t\wedge\tau_z^S} L_\tau d\tau}|Y_{t\wedge\tau_z^S}|^p\right] \leq |x_1 - x_2|^p. \tag{4.60}$$

Since the right-hand side of equation (4.60) is uniformly bounded with respect to $z > \bar{z}$, we can take the limit for $z \to \infty$ obtaining hence the inequality (4.55).■

4.5 Conclusions

In this chapter we have dealt with the well-posedness of the martingale problem, under very general assumptions on the coefficients of the related SDE. Such assumptions are satisfied by several models used in financial mathematics. Substantially, under local Lipschitz continuity assumptions and assuming the existence of a Lyapunov function, it is possible to prove the well-posedness of the martingale problem (see [17]). Furthemore, the boundary of the domain D cannot be reached in a finite time (or the process cannot blow up if the domain D is unbounded) by the process X, then it is constrained to stay in the interior of the domain. Then, starting from such results of existence and uniqueness of the solution, we have provided some interesting properties of continuity with respect to the initial data. In particular, in Theorem 4.4 we have proved a Lipschit-type continuity with respect to the initial datum until a stopping time. Moreover, we have provided a result of uniform continuity with respect to initial data for any time t, and a Hölder-type dependence of the expected value of the process with respect to time t.
Such results are not available in literature under our assumptions, but, as we have already stated, stronger hypoteses on the coefficients are usually admitted.

On the other hand, we have also provided a weighted continuity estimate. The weight function is directly connected with the coefficients. In particular it depends on the rate at which the coefficients μ and σ loose their Lipschitz continuity condition for the process approaching the boundary. Furthermore, in the special case of global Lipschitz continuous coefficients, our estimates turn to standard results.

On the other hand it is possible to argue that, if the probability that the process X reaches the boundary vanishes rapidly, this could have a regularizing effect on the dependence on the initial data. We are currently working to obtain some stronger results, in this direction.

Chapter 5

Regularity for singular risk-neutral valuation equation

In the previous chapter we have investigated the well-posedness of the martingale problem for $(\mathcal{A}, P_0)$. Furthermore the properties of the solution of the SDE (2.9) have been provided when Assumptions 4.1-4.3 are satisfied. Furthermore, we have stated and proved come properties of continuity for the stochastic process X_t^x with respect to the initial datum x and time t.

In this chapter we deal with the existence, uniqueness and regularity of the solution to the valuation equation (3.1), under the same assumptions made so far. In particular, we are able to prove that the regularity of $(x, t) \mapsto X_t^x$ is reflected in the solution $u(x, t)$ to the pricing equation.

It is common among practictioners, when one deals with problems in financial mathematics to develop numerical methods assuming existence, uniqueness and regularity of the solution to the valuation equation (3.1). However, when only Assumptions 4.1-4.4 are satisfied, the present literature is not able to grant the existence, uniqueness and regularity of the solution to the pricing problem (3.1). In the work presented by Costantini *et al.* in [17], some light has been put on this topic. Precisely, the authors proved that, when Assumptions 4.1-4.4 are satisfied, then the existence and uniqueness of a viscosity solution to the valuation problem (3.1) is granted. Unfortunalety, the viscosity solution theory ensures just continuity for the solution, but no additional regularity is *a priori* given. The aim of this chapter is exactly to give sufficient conditions in order to extend the results given in [17]. In particular, under some additional assumptions on the regularity of the final payoff and the running cost, it is possible to show that the viscosity solution gains some additional regularity.

This chapter is organized as follows. In the first section we rapidly introduce a very simple model where standard arguments given in the literature cannot be applied. In particular, we adapt the procedure proposed by Di Francesco, Pascucci and Polidoro in [22], under the framework of the space $\mathcal{S}^p$ and B-norm we have introduced in Chapter 3. We are able to show that such a simple problem have a unique classical solution under a suitable B norm. In the subsequent sections we consider the general problem (3.1). We give the main results provided in [17], that is under Assumptions 4.1-4.4 the general problem has a unique viscosity solution. These results are used to show that under additional structural condition for the operator $\mathcal{A}$, the viscosity solution is classical. In other words it is twice differentiable with respect to x, and once with respect to t.

5.1 Classical solution for pricing problems in a toy-model

Before considering the general problem, we want to introduce the issue of the existence of a regular solution to the valuation equation of the type (3.1), in a very simple case. In this case, we consider a time-reverted problem, in order to be compliant with the notation used in [22].
The problem we are considering is defined in a subset of $\mathbb{R}^2 \times \mathbb{R}$, with constant coefficients.

In this section, the notation $||\cdot||$ stands for sup norm, if not explicitly specified.

Consider a point $(\bar{x}, \bar{y}) \in \mathbb{R}^2$ and $\Delta > 0$. Let $H = D \times [0, T]$ be a regular domain where $D \subset \mathbb{R}^2$ is the ball centered in $(\bar{x}, \bar{y})$ with radius Δ. We suppose without loss of generality that $\bar{x} - \Delta > 0$. If it were not the case, we can always make a linear translation towards the origin. We want to deal with the following Cauchy-Dirichlet problem defined in H:

$$\begin{aligned} & u_t(x,y,t) - u_{xx}(x,y,t) + u_y(x,y,t) = f(x,y,t) \\ & u(x,y,t) = \phi(x,y,t) \qquad\qquad (x,y,t) \in \partial_P H \end{aligned} \tag{5.1}$$

where $\partial_P H$ is the parabolic boundary of H and the functions $\phi \in \mathcal{C}(H)$ and $f \in \mathcal{C}^\infty(H)$.

We notice that the problem (5.1) obviously verify Assumptions 4.1-4.4, then all the results in [17] can be applied. This ensure that the considered problem has a unique viscosity solution. On the other hand, now we want to study if such a kind of problem admits a solution that is more regular then the mere continuity. However, we have to consider that the diffusion matrix $a(x)$ is degenerate along one direction, and no Hörmander conditions are available. Hence, it is not clear if the considered operator has sufficient regularizing properies that guarantee the existence of a regular solution. Despite these facts, it is possible to show that the considered problem admits an unique classical solution in some way. Such a result is precisely stated in the following

Theorem 5.1. *Let H be the considered domain and $\mathcal{L}$ be the operator defined in the Cauchy problem (5.1). Then there exists a matrix B such that there exists a unique classical solution $u \in \mathcal{C}^{2,\alpha}_B(H(T)) \cap \mathcal{C}(\overline{H(T)})$ of the problem (5.1).*

We notice that the considered problem is very simple and have no application in modern financial mathematics, however, such a problem well introduces the issues we want to show in this chapter.

Before giving the proof of Theorem 5.1, we prove some partial results, that will be useful for our purposes.

Theorem 5.2. *Let $\mathcal{L}$ be a hypoelliptic differential operator defined on the compact set H and let G be the Green's function related to $\mathcal{L}$ with respect to the domain H. Let $\mathcal{J}, \mathcal{D} : \mathcal{C}^\infty(H) \to \mathcal{C}^\infty(H)$ be two linear operators such that, for each $h \in \mathcal{C}^\infty(H)$, they associate the functions:*

$$\mathcal{J}h = \int_H G(x,y,t,\xi,\eta,\tau)h(\xi,\eta,\tau)d\xi d\eta d\tau \tag{5.2}$$

$$\mathcal{D}h = \int_H G_y(x,y,t,\xi,\eta,\tau)h(\xi,\eta,\tau)d\xi d\eta d\tau, \tag{5.3}$$

then the following statements hold

i) $0 < ||\mathcal{J}|| < \infty$ *and* $0 < ||\mathcal{D}|| < \infty$

ii) There exists the operator $\mathcal{J}^{-1} : \mathcal{C}^\infty(H) \to \mathcal{C}^\infty(H)$, such that for each $v \in \mathcal{C}^\infty(H)$ it is hold true $\mathcal{J}^{-1}\mathcal{J}v = \mathcal{J}\mathcal{J}^{-1}v = v$.

iii) Ker $\mathcal{J} \subseteq$ Ker D.

Remark 5.1. *The existence of the Green's function G for the considered problem is granted by the results of Theorem 2.7 in [52].*

We observe that existence of the Green's function suggests that the solution to the problem (5.1) is regular.

Remark 5.2. *Given the properties of G it follows that the operators $\mathcal{J}$ and $\mathcal{D}$ are continuous.*

As a direct consequence of such a Remark we have the following

Corollary 5.1. *There exists a finite $\lambda > 0$, such that for each $h \in \mathcal{C}^\infty(H)$ it is hold true*

$$\big|\big|\mathcal{D}h\big|\big| \leq \lambda\big|\big|\mathcal{J}h\big|\big|. \tag{5.4}$$

Proof of Theorem 5.2 ***(i)*** **-** For simplicity of notation, in what follows, we denote by $(z,t) = (x,y,t)$ and $(\zeta,\tau) = (\xi,\eta,\tau)$ in H.

Since the function G is nonnegative, for each $h \in \mathcal{C}^\infty(H)$ where $||h|| = 1$, we have the following relations

$$\begin{aligned} ||\mathcal{J}|| &= \sup_{||h||=1} ||\mathcal{J}h|| = \sup_{||h||=1} \Big|\Big| \int_H G(z,t,\zeta,\tau)\, h(\zeta,\tau)\, d\zeta d\tau \Big|\Big| \\ &\leq \Big|\Big| \int_H G(z,t,\zeta,\tau)\, 1\, d\zeta d\tau \Big|\Big| \\ &= \Big|\Big| \int_H G(z,t,\zeta,\tau) d\zeta d\tau \Big|\Big| = ||v||, \end{aligned} \tag{5.5}$$

where v is the solution of the Cauchy problem

$$\begin{aligned} &\mathcal{L}v = -1 \\ &v = 0 \qquad\qquad (x,y,t) \in \partial_P H. \end{aligned} \tag{5.6}$$

Since the operator $\mathcal{L}$ is hypoelliptic and the sourcing term is smooth, then the solution v is smooth in a bounded domain, and then is bounded as well. This means that v verifies:

$$||v|| < \infty.$$

Furthermore, in force of the properties of the Green's function G we can write

$$||\mathcal{J}|| = \Big|\Big| \int_H G(z,t,\zeta,\tau) d\zeta d\tau \Big|\Big| > 0 \tag{5.7}$$

and then

$$0 < ||\mathcal{J}|| < \infty. \tag{5.8}$$

The proof for $\mathcal{D}$ is very close to the one we have provided for $\mathcal{J}$, but an additional consideration is needed. In particular in force of the fact that v is smooth, then $||\mathcal{D}|| \leq ||v_y||$. Hence we can end the proof of the first item of the Theorem.

(ii) - We can give an explicit representation of the operator $\mathcal{J}^{-1}$. Indeed, let $\mathcal{L}$ be the considered differential operator. We define $\mathcal{J}^{-1}$ in the following way:

$$\mathcal{J}^{-1} = \mathcal{L}. \tag{5.9}$$

The thesis comes directly from the properties of the Green's function. Indeed, let $v \in$

$\mathcal{C}^{\infty}(H)$, then we can write the following relations:

$$\begin{aligned}\mathcal{J}^{-1}\mathcal{J}v &= \mathcal{L}\int_H G(z,t,\zeta,\tau)v(\zeta,\tau)d\zeta d\tau \\ &= \int_H \mathcal{L}G(z,t,\zeta,\tau)v(\zeta,\tau)d\zeta d\tau \\ &= \int_H \delta(z-\zeta,t-\tau)v(\zeta,\tau)d\zeta d\tau \\ &= v \end{aligned} \tag{5.10}$$

and then we have the thesis.

(iii) - Consider $h \in \mathcal{C}^{\infty}(H) \cap \text{Ker } \mathcal{J}$. We want to verify that $h \in \text{Ker } \mathcal{D}$.

Let $\mathcal{L}$ be the considered differential operatorator, and let v be the solution of the Cauchy problem:

$$\begin{aligned}&\mathcal{L}v = -h \\ &v = 0 \qquad (x,y,t) \in \partial_P H\end{aligned}$$

then, by the properties of the Green's function G it is hold true $v = \mathcal{J}h$. Furthermore, since $h \in \text{Ker } \mathcal{J}$ by hypotesis, it is hold true $v \equiv 0$ in all H, up to the boundary. Then we can write:

$$\begin{aligned}||\mathcal{D}h|| &= \Big|\Big|\int_H G_y(z,t,\zeta,\tau)\,h(\zeta,\tau)\,d\zeta d\tau\Big|\Big| \\ &= \Big|\Big|\partial_y \int_H G(z,t,\zeta,\tau)\,h(\zeta,\tau)\,d\zeta d\tau\Big|\Big| \\ &= \big|\big|\partial_y v\big|\big|. \end{aligned} \tag{5.11}$$

By the definition of v, for each $(x,y,t) \in \overline{H}$, the relation $\partial_y v \equiv 0$ has to be verified, indeed by the regularity if v such a relation has to be hold true for each $(x,y,t) \in H \setminus \partial H$. A priori we cannot consider directly the continuity of the derivative since up the boundary just the continuity of the solution is ensured, then, we consider the differential ratio

$$\frac{v(x,y,t) - v(x,\tilde{y},t)}{y - \tilde{y}} \tag{5.12}$$

with $\tilde{y}$ is any point in ∂H. We know that the differential ratio in (5.12) is well defined and is identically zero. Then there exists $\limsup$ for $y \to \tilde{y}$ and it is hold true

$$\limsup_{y \to \bar{y}} \Big|\frac{v(x,y,t) - v(x,\bar{y},t)}{y - \bar{y}}\Big| = 0. \tag{5.13}$$

Then we can write

$$||\mathcal{D}h|| = \big|\big|\partial_y v\big|\big| = 0 \tag{5.14}$$

that implies the item *(iii)* and ends the proof of Theorem 5.2.■

Now the proof of Corollary 5.1 is automatically given.

Proof of Corollary 5.1 If $h \in$ Ker $\mathcal{J}$ there is nothing to prove. Then we suppose that $h \in \mathcal{C}^\infty(H) \cap (\text{Ker } \mathcal{J})^c$. By Theorem 5.2 and Remark 5.2 the following relations are hold true:

$$\big|\big|\mathcal{D}h\big|\big| = \big|\big|\mathcal{D}\mathcal{J}^{-1}\mathcal{J}h\big|\big| \leq \big|\big|\mathcal{D}\mathcal{J}^{-1}\big|\big| \cdot \big|\big|\mathcal{J}h\big|\big|. \tag{5.15}$$

Consider $h^* \in \mathcal{C}^\infty(H)$ such that, for $h \in \mathcal{C}^\infty(H)$ we have

$$||\mathcal{D}\mathcal{J}^{-1}h^*|| = \sup_{||h=1||} ||\mathcal{D}\mathcal{J}^{-1}h||. \tag{5.16}$$

Hence we can write

$$\begin{aligned} ||\mathcal{D}\mathcal{J}^{-1}h^*|| &= \left|\left|\partial_y \int_H G(x,y,t,\xi,\eta,\tau)\mathcal{L}h^*(\xi,\eta,\tau)d\xi d\eta d\tau\right|\right| \\ &\leq \lambda, \end{aligned} \tag{5.17}$$

with $0 < \lambda < \infty$ in force of the properties of the Green's function. Then, we can write the relation (5.15) as

$$\big|\big|\mathcal{D}h\big|\big| \leq \lambda \cdot \big|\big|\mathcal{J}h\big|\big|. \blacksquare \tag{5.18}$$

Before studying the regularity of the solution of the Cauchy problem (5.1) we make the following:

Remark 5.3. *Let $\mathcal{L}$ be the operator verifying the hypothesis of Theorem 5.2. Then Theorem 5.2 and Corollary 5.1 are hold true also for the operator $\mathcal{L}_\lambda$ defined as $\mathcal{L}_\lambda = \mathcal{L} - \lambda$, with the same coefficient λ.*

The proof comes directly by substitution of h by $h \cdot e^{-\lambda t}$. ■

Proof of Theorem 5.1 The core of the proof is substantially the same given in [22]. We add and subtract the term $\frac{x}{x_0}u_y(x,y,t)$ to the Cauchy problem (5.1). Rearranging the terms we get:

$$\begin{aligned} &u_t - u_{xx} + \frac{x}{x_0}u_y = f + \left(\frac{x}{x_0} - 1\right)u_y \\ &u(x,y,t) = \phi(x,y,t) \qquad\qquad (x,y,t) \in \partial_P H \end{aligned} \tag{5.19}$$

where x_0 is a costant coefficient, that depends on the considered domain, and is such that $x_0 = \bar{x} - (\Delta + \epsilon)$ where ϵ is chosen small enough that $x/x_0 > 0$, for any $x \in D$.

For each $n \geq 1$, we consider the sequence $u^{(n)}$ of the solution of the problem, defined as

$$u_t^{(n)} - u_{xx}^{(n)} + \frac{x}{x_0} u_y^{(n)} = g^{(n-1)} \tag{5.20}$$
$$u^{(n)}(x,y,t) = \phi(x,y,t) \qquad (x,y,t) \in \partial_P H$$

where $g^{(n)}(x,y,t,u_y^{(n)}) = f + \big(\frac{x}{x_0} - 1\big)u_y^{(n)}$, and $u^{(0)}(z) \in \mathcal{C}^\infty(H)$ is defined as:

$$u^{(0)}(x,y,t) = e^{\gamma(T-t)} + ||\phi||_\infty - 1 \tag{5.21}$$

with $\gamma = ||\phi||_\infty + ||f||_\infty$.

Remark 5.4. *A direct calculation shows that $g^{(0)} - \mathcal{L}_0 u^{(0)} \geq 0$ if $(x,y,t) \in H$ and $u^{(0)} \geq \phi$ for $(x,y,t) \in \partial_P H$, where $\mathcal{L}_0$ is the operator of the problem (5.20).*

Remark 5.5. *For each $n \in \mathbb{N}$ the solution $u^{(n)} \in \mathcal{C}^\infty(H)$. Indeed the operator $\mathcal{L}_0$ defined as $\mathcal{L}_0 = \partial_t - \partial_{xx} + \frac{x}{x_0}\partial_y$ is hypoelliptic since all the coefficients are smooth anth the Kalman's condition is verified, then the Hörmander condition is hold true, and the sourcing term is in $\mathcal{C}^\infty(H)$.*

We observe that the existence of the solution to the Cauchy problem (5.20) is granted by the results in [22]. By adapting the technique used by Pascucci, we can show that the sequence $u^{(n)}$ converges towards a function in $\mathcal{C}_B^{2,\alpha}(H)$, where $\mathbf{B}$ is the matrix given by

$$\mathbf{B} = \begin{pmatrix} 0 & 0 \\ \frac{1}{x_0} & 0 \end{pmatrix}.$$

First of all we want to show by induction that the solutions $\big(u^{(n)}\big)_n$ is a decreasing sequence, uniformly bounded and solve the Cauchy problem (5.20) (this is true by construction). Then it is possible to show that, in the case we are considering now, the solutions seqence of solutions $\{u^{(n)}\}_n$ have the same properties of regularity of the sourcing term given at $n-1$. Indeed the sourcing term of the problem n-th depends on the solution at step $(n-1)$.

We consider the following Cauchy problem:

$$u_t^{(n)} - u_{xx}^{(n)} + \frac{x}{x_0} u_y^{(n)} - \lambda u^{(n)} = f + \Big(\frac{x}{x_0} - 1\Big) u_y^{(n-1)} - \lambda u^{(n-1)} \tag{5.22}$$
$$u^{(n)}(x,y,t) = \phi(x,y,t) \qquad (x,y,t) \in \partial_P H$$

We suppose that for each $n \in \mathbb{N}$ the following inequalities are hold true:

$$-u^{(0)} \leq u^{(n+1)} \leq u^{(n)} \leq u^{(0)} \tag{5.23}$$

where $u^{(0)}$ is defined in (5.21). This fact implies that the sequence $u^{(n)}$ converges in $\overline{H(T)}$. We indicate by u such a limit.

Since $u^{(n)}$ is the solution of the Cauchy problem (5.22), and converges to u, we can follow the same arguments used in [22] in order to state that the sequence $u^{(n)}$ admits a subsequence $u^{(n_j)}$ that locally converges in $\mathcal{C}_B^{2,\alpha}(K)$, where K is any compact subset of $H(T)$.

At the end, we can conclude this part of the proof by taking the limit for n_j towards infinity, and then we get

$$u_t - u_{xx} + u_y = f \tag{5.24}$$

whit final condition $u|_{\partial_P H} = \phi$.

In order to conclude this part of the theorem we have to prove the continuity of the solution up to boundary. This fact is ensured by the existence of the barrier function on the boundary of a regular domain. The precise procedure can be found, for example in [22] and [56]. In particular, the existence of such a barrier function is granted by the considered operator $\mathcal{L}_0$ and the regularity of the domain (see Definition 5.3 and Remark 5.7 below) and is explicitly provided in [52].

Now, if we show that the relation (5.23) holds true for every $n \geq 1$ then we have proved Theorem 5.1.

In force of Remark 5.4, the condition $u^{(1)} \leq u^{(0)}$ is obviously satisfied. Hence, let $\mathcal{L}_0$ and $g^{(n)}$ be respectively the operator and the sourcing term, defined in the Cauchy problem (5.20). By the result of the problem (5.22) and Remark 5.4 we can write:

$$\mathcal{L}_0\big(u^{(1)} - u^{(0)}\big) - \lambda\big(u^{(1)} - u^{(0)}\big) = g^{(0)} - \mathcal{L}_0 u^{(0)} \geq 0. \tag{5.25}$$

Since $u^{(1)} \leq u^{(0)}$ on $\partial_P H$, by the maximum principle for cylindrical domains (see for instance Proposition 2.2 in [52]), the condition $u^{(1)} \leq u^{(0)}$ is satisfied on all $\overline{H(T)}$.

As shown in [22], we show the relation (5.23) by induction. Suppose the relation $u^{(n)} \leq u^{(n-1)}$ be hold true until n, we show that it is hold true also $u^{(n+1)} \leq u^{(n)}$. This fact is true since $u^{(n+1)}$ and $u^{(n)}$ verify the same boundary conditions. Furthermore we have

$$\begin{aligned} \mathcal{L}_0\big(u^{(n+1)} - u^{(n)}\big) - \lambda\big(u^{(n+1)} - u^{(n)}\big) &= \Big[\Big(\frac{x}{x_0} - 1\Big)\Big(u_y^{(n)} - u_y^{(n-1)}\Big)\Big] \\ &- \lambda\big(u^{(n)} - u^{(n-1)}\big). \end{aligned}$$

By Corollary 5.1 and by the definition of x_0, we can get directly the relation

$$\begin{aligned}\mathcal{L}_0\big(u^{(n+1)} - u^{(n)}\big) - \lambda\big(u^{(n+1)} - u^{(n)}\big) &\geq -\lambda\frac{x}{x_0}\big(u^{(n)} - u^{(n-1)}\big)\\ &\geq 0. \end{aligned} \tag{5.26}$$

Hence, in force of the maximum principle for cylindrical domains we can state that $u^{(n+1)} \leq u^{(n)}$ on $\partial_P H$. At the hand, since $u^{(1)} \leq u^{(0)}$ is satisfied as well, then the relation $u^{(n)} \leq u^{(n-1)}$ holds true for each n on all the domain $H(T)$.

As remarked in [22], the same arguments show that for each n it is hold true also $-u^{(0)} \leq u^{(n)}$, that ends the proof of Theorem 5.1. ■

Remark 5.6. *We notice also that the way we have followed in order to prove the existence of a classical solution to the problem (5.1) provide a recursive semianalytical expression of the solution for the Cauchy problem (5.1).*

5.2 Existence and uniqueness of the solution for the general valuation equation

In the previous section we have considered a simple model where the infinitesimal generator is very similar to the ones considered in several works by Lanconelli, Pascucci, Polidoro. Unfortunately, in their works, the authors consider operators with a specific form and, in several cases of interest in financial applications their results cannot be directly applied. In particular, when only Assumptions 4.1-4.4 are verified, the theory of hypoelliptic operators and viscosity solution theory cannot be applied. On the other hand, by adapting the technique used by Lanconelli and Pascucci for the case of hyoelliptic operators, we can prove the existence of a unique solution that is also regular in some sense. In the following sections we shall introduce the problem and main results related to the solution of the valuation equation.

We have seen in the previous chapter that the presence of the Lyapunov function $V(x)$ ensures the well-posedness of the martingale problem $(\mathcal{A}, P_0)$ for any initial distribution P_0. Furthermore, the presence of such a Lyapunov function avoid the process X to reach the boundary of the domain, or blows up in a finite time. Such a result gives a precise formalization to the common feeling that, for some specific behaviour of the coefficients, there are some regions of the space D that are forbidden to the process X. On the other hand the well-known Feller's condition can be seen, in some ways, as a particular case of this result. Expecially, we see in Chapter 6 that, in the case of the CIR model, the Feller's condition is enough to ensure a Lyapunov-type condition verifying Assumption 4.3.

On the other hand the results provided in [17] deal with the well-posedness of the pricing problem (3.1). In particular the authors prove that when Assumptions 4.1-4.4 are fulfilled, then the pricing problem (3.1) admits one and only one viscosity solution. These results are presented in the following two theorems.

Theorem 5.3 (Theorem 2.9 in [17]). *For every $x \in D$, let X^x be the process of Theorem 4.3 with initial condition $P_0 = \delta(x)$. Then, for every $t \in [0, T]$*

$$\mathbb{E}\left[\left|\phi(X^x_{T-t})e^{-\int_0^{T-t} c(X^x_r)dr} - \int_0^{T-t} f(X^x_s, t+s)e^{-\int_0^s c(X^x_r)dr}ds\right|\right] < \infty. \tag{5.27}$$

For each $(x, t) \in D \times [0, T]$, the function $u(x, t)$ defined as

$$u(x,t) = \mathbb{E}\left[\phi(X^x_{T-t})e^{-\int_0^{T-t} c(X^x_r)dr} - \int_0^{T-t} f(X^x_s, t+s)e^{-\int_0^s c(X^x_r)dr}ds\right] \tag{5.28}$$

is continuous on $D \times [0, T]$. Furthermore it is a viscosity solution of the problem (3.1) and satisfies the relation

$$|u(x,t)|\varphi(|u(x,t)|) \leq C_T(1 + V(x)) \tag{5.29}$$

for every $(x, t) \in D \times [0, T]$.

We remark that the relation (5.29) is a direct consequence of Assumption 4.4. Such a condition is needed in order to guarantee the uniqueness of the viscosity solution. On the other hand, it will be clear in Chapter 6 that in several models, such as in stochastic volatility models, the existence and uniqueness may not hold true. This is the case, for instance, when the coefficients of the SDE and the final payoff, or f, have high growth rate. Assumption 4.4 provides a bound on the growth rate allowed to the final payoff and f. As expected this type of bound is also fulfilled by the solution $u(x, t)$ through the inquality (5.29).

Theorem 5.4 (Theorem 2.10, in [17]). *There exists only one viscosity solution to the problem (3.1) satisfying (5.29).*

The relation (5.27) is a direct consequence of Assumption 4.4 on ϕ, f and c, and the result of Theorem 4.3, relation (4.23). We notice that, by the same reason it is possible to prove the boundedness of the terms in the inequality (5.28) separately. This result implies that the function $u(x, t)$ given in (5.28) is well defined.

The authors get proof of the continuity of the function $u(x, t)$ and the fact that it is a viscosity solution is achieved in two distinct steps. In particular, the continuity of the function $u(x, t)$ is shown taking a sequence $\{x_k\}_k$ converging to $x \in D$, and by observing that the process X^{x_k} converges weakely to X^x (see for example Theorem 8.15, Chapter 4, in [28]). Therefore, by setting

$$Y^x_t = e^{-\int_0^t c(X^x_s)ds},$$

since $\mathbb{P}(X_t^x \neq X_{t^-}^x) = 0$ for every $t \geq 0$ it is possible to state that for every $\{(x_k, t_k)\} \subseteq D \times [0,T]$ converging to $(x,t) \in D \times [0,T]$,

$$\phi(X_{T-t_k}^{x_k})Y^{x_k} - \int_0^{T-t_k} f(X_s^{x_k}, s+t)Y_s^{x_k} ds \xrightarrow{\mathcal{L}} \phi(X_{T-t}^x)Y^x - \int_0^{T-t} f(X_s^x, s+t)Y_s^x ds. \tag{5.30}$$

The assertion arises by observing that the random variables in the left-hand side of (5.30) are uniformly integrable due to the relation (5.27), which allows us to take expectation in (5.30). Then the expectation of the-left hand side of (5.30) converges to the expectation of the right-hand side.

The proof that the function u defined in (5.29) is achieved by showing that it is both sub and super-solution in the viscosity sense for the problem (3.1). In doing that, the fact that X^x is a strong Markov process and verifies the martingale problem for $(\mathcal{A}, \delta(x))$ plays a crucial role.

Therefore, Assumptions 4.1-4.4 are sufficient conditions in order to guarantee that there exists a viscosity solution to the pricing problem (3.1). As usual in the viscosity solution theory, the uniqueness of such a solution follows directly from the comparison principle below.

Theorem 5.5 (Theorem 4.6, in [17])**.** *Let $\underline{u}$ and $\overline{u}$ be respectively a viscosity sub-/super-solution of (3.1) both satisfying (5.29). Then $\underline{u}(x,t) \leq \overline{u}(x,t)$ for all $(x,t) \in D \times [0,T]$.*

In this case the authors build a penalization function $w_\beta(t,x,y)$ defined as

$$w_\beta(t,x,y) = \underline{u}(x,t) - \overline{u}(y,t) - \beta\left[1 + V(x) + V(y)\right], \tag{5.31}$$

where $\beta > 0$ is a costant arbitrarly chosen, $x, y \in D$, the functions $\underline{u}$ and $\overline{u}$ are respectively sub/super-solution to the pricing problem (3.1) and $V(x)$ is the Lyapunov function of Assumption 4.3.

In force of the relation (5.29), for each β there exists a suitable compact subset K_β such that for each $(x,y) \in K_\beta$ and $t \in [0,T]$ it is verified $w_\beta(t,x,y) \leq 0$.

Then, for $t \in [0,T]$, the auxiliary function

$$\vartheta_\beta(t) = \lim_{r \to 0^+} \sup\{\max(w_\beta(t,x,y), 0) : x, y \in D, |x-y| < r\}, \tag{5.32}$$

is defined. If we denote by ϑ_β^* its upper semi-continuous envelope, the thesis is then proved by the authors showing that for each $t \in [0,T]$ and $\beta > 0$, it holds

$$\vartheta_\beta^*(t) = 0. \tag{5.33}$$

5.3 Classical solution to the pricing problem

In this section we see that the results we have got untill now can be used directly in order to show that, under suitable conditions on the diffusion matrix $a(x) = \sigma(x)\sigma(x)^\top$, there exists a unique classical solution of the pricing problem (3.1) in $\mathcal{C}^{2,1}$ in some open bounded subset S included in D. This result is quite standard in the theory of the parabolic equations. However, we are able to prove that such a solution coincides with the unique viscosity solution found in Theorems 5.3 and Theorem 5.4 in the domain S. This result comes directly from our knowledge on the existence and uniqueness of a global viscosity solution. In particular, we are able to restrict ourslves to a bounded domain and to require the standard properties of regularity be valid just locally.
In other words, under the same hypoteses of Theorem 5.3 and Theorem 5.4, strongening the assumptions on the diffusion matrix $a(x)$, and the functions $c(x)$ and $f(x,t)$ on some bounded domain S, the viscosity solution is not a mere continuous function or Hölder continuous, but in S is twice differentiable with respect to x and once with respect to t.

In the first part of the section we recall a fundamental result on the parabolic partial differential equations for pure diffusive operators $\mathcal{A}_d$ as defined in the relation (3.3). In the second part of the section we give our main result dealing with the additional regularity for the viscosity solution $u(x,t)$.

In what follows we consider an open bounded ball S that is subset of D, such that the closure of S is included in D. We remark that S has boundary $\mathcal{C}^2$. When we deal with bounded domains, and the integro-differential is pure diffusive, then it is well known that the continuity of the solution on the boundary is ensured if a suitable barrier function is defined on the domain, whenever such a domain is regular enough, and the initial data are continuous (see e.g. [56] and the technique used by [52], or the results in [9]). The regularity required for such a domain depends on the form of the differential operator $\mathcal{A}_d$.

Definition 5.1. *Let $S \subset \mathbb{R}^n$ be a bounded and closed domain. Then S is regular if, for each $x_0 \in \partial S$, there exist un open boundary Ω of x_0 and a function $f : \Omega \to \mathbb{R}$, in $C^1(\Omega)$ such that $\nabla f(x) \neq 0$ in Ω and:*

$$\partial S \cap \Omega = \{x \in \Omega : f(x) = 0\} \tag{5.34}$$

$$\textit{int}\ S \cap \Omega = \{x \in \Omega : f(x) < 0\} \tag{5.35}$$

Definition 5.2. *Let S be a regular domain of $\mathbb{R}^n$ and $x_0 \in \partial S$. Let us define $\zeta(x_0)$ the outer normal vector to S in x_0 the versor defined as:*

$$\zeta(x_0) = \left.\frac{\nabla f(x)}{|\nabla f(x)|}\right|_{x_0} \tag{5.36}$$

We notice that the adjective outer is motivated by the fact that, it is possible to show that, the versor defined in (5.36) is directed out of the domain S.

Now we can define the properties of the domain where the existence of a barrier function to $\mathcal{L}$ is ensured.

Definition 5.3. *Let S be an open subset of D. The point x_0 in ∂S is strongly $\mathcal{A}_d$-regular if there exists a $\mathcal{A}_d$-non-charachteristic outer normal to S in x_0, i.e. a vector $\zeta \neq 0$ such that $B_{|\zeta|}(x_0 + \zeta) \cap S = \{\emptyset\}$ and $\langle a(x_0, t)\zeta, \zeta\rangle > 0$ for every $t \in [0, T]$.*

Remark 5.7. *If the diffusion matrix $a(x)$ is uniformly positive defined in the domain S, and S is regular in the sense of the Definition 5.1, then for each x_0 in ∂S is strongly $\mathcal{A}_d$-regular.*

The Remark 5.7 can be weakened to any domain $\tilde{S}$ such that the following hypotheses are hold true

i) $\partial\tilde{S}$ is differentiable almost everywhere, at least in a finite countable subset X_0 of $\partial\tilde{S}$,

ii) there exists $\tilde{S}^*$ such that $\partial\tilde{S}^*$ is differentiable and $\tilde{S} \subset \tilde{S}^*$,

iii) for each $x_0 \in X_0$ it is valid $x_0 \in \partial\tilde{S}^*$.

Such hypotheses include, obviously, the domain considered by Lanonelli and Pascucci in [52] and Di Francesco, Pascucci, Polidoro in [22]. Instead, a very simple counter example is given by a cross domain. In such a case, indeed, due to the intersection of the two axes of the cross, it is not possible to fulfill at the same time both the points *(ii)* and *(iii)*.

Proof of Remark 5.7 The proof of the Remark comes directly by standard arguments. First of all we notice that since the diffusion matrix $a(x)$ is uniformly positive define in S, then the condition $\langle a(x)\zeta, \zeta\rangle > 0$ for each $\zeta \neq 0$.
Now we have to show that for each $x_0 \in \partial S$ there exists a vector $\zeta \neq 0$ outer normal to S in x_0 such that $B_{|\zeta|}(x_0 + \zeta) \cap S = \{\emptyset\}$, with S is a regular domain in the sense of the Definition 5.1.

Consider $x_0 \in \partial S$ and $\zeta(x_0)$ the outer normal vector to S in x_0. Let v be defined as $v = x_0 + \lambda\zeta(x_0)$ with $\lambda > 0$. For λ arbitrarily small, the point v does not belong to S, indeed, if λ is small enough v belongs to Ω of the definition 5.1. Furthermore we can write

$$\frac{d}{d\lambda} f\big(x_0 + \lambda\zeta(x_0)\big)\Big|_{\lambda=0} = \langle\nabla f(x_0), \zeta(x_0)\rangle = \big|\nabla f(x_0)\big| > 0. \tag{5.37}$$

Due to the regularity of the function f, there exists $\epsilon > 0$ such that $d/d\lambda f\big(x_0+\lambda\zeta(x_0)\big) > 0$ for each $\lambda \in [0, \epsilon)$. Furthermore, since $f(x_0) = 0$ we have

$$f\big(x_0 + \lambda\zeta(x_0)\big) > 0 \qquad \forall \in (0, \epsilon]. \tag{5.38}$$

Since the domain S is regular, for each $x_0 \in \partial S$ there exists a neighbour I of x_0 such that, for each $x \in I \cap \partial S$, the relation (5.38) is hold true, for ϵ depends on the point.
Then, by the continuity of f, there exists a domain $\tilde{\Omega} \supset\supset \Omega$ such that $f(x) > 0$ vof each $x \in \tilde{\Omega} \cap \Omega^c$.
We indicate by $\bar{\rho} = \min_{\rho} d(\partial\tilde{\Omega}, \partial\Omega)$ the minimum dinstance between $\partial\tilde{\Omega}$ and $\partial\Omega$. This means that for each $x_0 \in \partial\Omega$ there exists $\rho > 0$ such that $f(x) > 0$ for each $x \in B_{\rho}\big(x_0 + \rho v(x_0)\big)$, that ends the proof. ■

Consider the operator $\mathcal{A}_d$ restricted to the domain $\overline{S} \times [0,T]$. Then the coefficients $a(x)$ and $\mu(x)$ of $\mathcal{A}_d$ are Lipschitz continuous in $\overline{S} \times [0,T]$. Now we make the additional

Assumption 5.1. *Let S be a regular bounded open domain. Let $\mathcal{A}$ be a pure diffusive operator with coefficients μ and a restricted to $\overline{S} \times [0,T]$. Let a be the diffusion matrix of the operator $\mathcal{A}$ such that for each $x \in \overline{S}$ and $\zeta \in \mathbb{R}^d$ there exists $\lambda > 0$ such that*

$$\langle a(x)\zeta, \zeta\rangle \geq \lambda|\zeta|^2. \tag{5.39}$$

Suppose $c(x)$ and $f(x,t)$ be Hölder continuous in $\bar{S} \times [0,T]$.

Remark 5.8. *It is trivial to show that every point x_0 in the boundary of the considered ball S is strongly $\mathcal{A}_d$-regular in the sense of the Definition 5.3.*

If all Assumption 4.1 and Assumption 5.1 are verified, then the following result is valid

Theorem 5.6 (Theorem 3.6 p. 138, [35])**.** *Let ϕ and g be a continuous function respectively on $\overline{S}$, and $\partial S \times [0,T]$ such that $g(x,T) = \phi(x)$. Assume also that there exists a barrier function at every point of S. Then there exists a unique solution $v \in \mathcal{C}^{2,1}(S \times [0,T)) \cap \mathcal{C}(\overline{S} \times [0,T])$ to the problem*

$$(CP)\begin{cases} \partial_t v + \mathcal{A}_d v - cv = f & (x,t) \in S \times [0,T) \\ v(x,T) = \phi(x) & x \in S \\ v(x,t) = g(x,t) & (x,t) \in \partial S \times [0,T]. \end{cases}$$

Remark 5.9. *In force of the Remark 5.7, each point of the domain ∂S is $\mathcal{A}_d$-strongly regular, then the barrier function of Theorem 5.6 is explicitely provided in [52].*

In what follows we suppose that for each set S such that $\bar{S} \subset D$ Assumption 4.1-Assumption 4.4, and Assumption 5.1 are hold true.

Remark 5.10. *Under the considered Assumptions there exists a unique viscosity solution $u(x,t)$ to the pricing problem (3.1), where the operator $\mathcal{A}$ in the problem (3.1) is a pure diffusive problem.*

We remark furthermore that the problem (3.1) in D the does not need the boundary conditions.

Now we state the following

Theorem 5.7. *Consider the domain S as previously defined and let $\mathcal{A}$ be a pure diffusive operator. Suppose Assumptions 4.1-Assumption 4.4 and Assumption 5.1 are hold true. Then, there exists a unique classical solution $v \in \mathcal{C}^{2,1}(S \times [0,T)) \cap \mathcal{C}(\overline{S} \times [0,T])$ to the problem*

$$(CDP) \begin{cases} \partial_t v + \mathcal{A}_d v - cv = f & (x,t) \in S \times [0,T) \\ v(x,T) = u(x,T) & x \in S \\ v(x,t) = u(x,t) & (x,t) \in \partial S \times [0,T] \end{cases}$$

where u is defined in (5.28) and is the viscosity solution of the pricing problem (3.1).

An analogous Theorem is provided By Baldi in [9] (Theorem 9.5, p.193) that consider a domain Q with $\partial Q \in \mathcal{C}^2$. Such Assumption is strong enough to ensure the existence of a barrier function.

Proposition 5.1. *Under the hypotheses of Theorem 5.7 the classical solution v to the pure diffusive problem (CDP) coincide with viscosity solution u to the problem (3.1).*

Theorem 5.7 is telling us that there exists a unique classical solution $v(x,t)$ to a problem where the infinitesimal generator $\mathcal{A}_d$ is the diffusive part of the operator (3.2), the sourcing term is the same of the problem (3.1), and the function $v(x,t)$ coincides, at the parabolic boundary, with the viscosity solution $u(x,t)$ of the valuation equation (3.1).

Proposition 5.1, instead, says that such a classical solution $v(x,t)$ coincides with $u(x,t)$ in all the considered cylindrical set $S \times [0,T]$.

Proof of Theorem 5.7 The existence of a unique viscosity solution $u(x,t)$ to the problem (3.1) is ensured in force of Theorem 5.3 and Theorem 5.4.

All the hypotheses of Theorem 5.6 are fulfilled, indeed the existence of a barrier function is ensured by the regularity of the considered domain, the coefficients are Lipschitz continuous in $\bar{S}$, the diffusion matrix $a(x)$ is uniformly positive defined in $\bar{S}$, and the terms c and f are Hölder continuous by hypoteses. Then, applying Theorem 5.6, we have that there exists a unique classical solution $v(x,t) \in \mathcal{C}^{2,1}(S \times [0,T)) \cap \mathcal{C}(\overline{S} \times [0,T])$ to the pricing problem (CDP) that coincides with the viscosity solution of the problem (3.1) at the boundary of $S \times [0,T]$. ■

If we show that the solution $v(x,t)$ coincides with $u(x,t)$ in all the domain S, then we have got the result.

Since the infinitesimal generators $\mathcal{A}$ and $\mathcal{A}_d$ for u and v does not coincide, it is not possible to show that u coincides with v in S by using standard arguments, but we need some partial results.

Proof of Proposition 5.1. Now we can prove that the solution $v(x,t)$ of the Cauchy-Dirichlet (CDP) problem coincides with the solution $u(x,t)$ to problem (3.1) in the cylindrical domain $H = S \times [0,T]$. Such a result is achived directly from the properties of the solution $u(x,t)$. We observe that, using the same approach, the same result is given in [23] for a particular form of the integro-differential operator.

We know that there exists a unique viscosity solution $u(x,t)$ to the pricing problem (3.1) in the domain D and a unique classical solution to the problem (CDP). For a fixed point $(\bar{x},\bar{t}) \in (0,T) \times D$ and $\delta > 0$, consider the problem

$$(CDPv) \begin{cases} \partial_t v + \mathcal{A}_d v - cv = f & (x,t) \in S \times (\bar{t}, \bar{t}+\delta) \\ v = u & (x,t) \in \partial_P [S \times (\bar{t}, \bar{t}+\delta)]. \end{cases}$$

Since the quantity

$$u\left(X^{\bar{x}}_{t\wedge\tau}, \bar{t} + t\wedge\tau\right) e^{-\int_0^{t\wedge\tau} c(X^{\bar{x}}_s)ds} - \int_0^{t\wedge\tau} f(X^{\bar{x}}_s, s) e^{-\int_0^s c(X^{\bar{x}}_z)dz} ds$$

is an $\{\mathcal{F}^X_t\}$-martingale for every $\{\mathcal{F}^X_t\}$-stopping time τ, we consider the first exit time τ of $(X^{\bar{x}}_{s-\bar{t}})_{s\geq\bar{t}}$ from the domain S. Furthermore, since u and v coincide on the parabolic boundary of $S \times (0,T)$, and u is a martingale we can write

$$u(\bar{x},\bar{t}) = \mathbb{E}\left[v(X^{\bar{x}}_{\tau-\bar{t}}, \tau) e^{-\int_0^\tau c(X^{\bar{x}}_s)ds} - \int_0^\tau f(X^{\bar{x}}_s, s) e^{-\int_0^s c(X^{\bar{x}}_z)dz} ds\right], \tag{5.40}$$

and then, by applying Itô's lemma to the right side of the equation (5.40) we get

$$\begin{aligned} u(\bar{x},\bar{t}) &= \mathbb{E}\left[v(X^{\bar{x}}_{\tau-\bar{t}}, \tau) e^{-\int_0^\tau c(X^{\bar{x}}_s)ds} - \int_0^\tau f(X^{\bar{x}}_s, s) e^{-\int_0^s c(X^{\bar{x}}_z)dz} ds\right] \\ &= v(\bar{x},\bar{t}). \end{aligned} \tag{5.41}$$

■

Notice that the result we have got in Theorem 5.7 and in Proposition 5.1 is independent on the center of the ball S and radius. This means that the properties we have found so far are locally valid, and may be applied in each ball S of D. Iterating the arguments for each $S_n \subset D$, we can state that the viscosity solution u of the problem (3.1) is also $\mathcal{C}^{2,1}$ in each compact $D_0 = \cup_n S_n$ such that $D_0 \cap \partial D = \{\emptyset\}$. In particular this is true since the existence of a unique viscosity solution to the problem (3.1) is a global property of the problem. Then, once that the existence of such a solution is proven, additional properties of regularity are local features of the solution.

This strategy allows us to define a solution $u(x,t)$ that is *a priori* merely continuous and solves the valuation problem in the viscosity sense. However, there may exist sub-domains $S_k \subset D$ for $k = 1, 2, ...$ such that, the solution $u(x,t)$ gains more regularity.

5.4 Conclusions

In this chapter we have considered a very simple model in order to introduce the problem of the well-posedness of the pricing equation (3.1). However, we have seen that it is possible to give some results on the existence and uniquenss of a regular solution. On the other hand, a very general result provided by [17] is shown. In such a work under Assumptions 4.1-4.4 the existence of a unique viscosity solution is proved. Unfortunately the viscosity solution theory ensures just that the solutions are continuous but no further regularity is granted a priori.

However, we have proved that, if there exists a domain where the diffusive matrix is uniformly positive defined up to the boundary and the running cost and the interest rate are Hölder continuous, then the viscosity solution is granted to be twice differentiable with respect to x and once with respect to t.

We conjecture that this fact is a general property of the considered problem, given the particular expression of the viscosity solution $u(x,t)$. Then whenever there exists a unique viscosity solution, further regularity could be investigated just locally. This fact should allows us to restrict ourselves to subdomain, and then we are not forced to require that some of the assumptions are satisfied in all the considered domain.

Chapter 6

Application to jump-diffusion stochastic volatility models

In the previous chapters we have seen that many problem of interest for finance are quite stiff to be dealt with a rigorous mathematical approach, and in some cases the present literature cannot be applied. We referer in particular to the cases where the stochastic process is allowed to have sudden jumps. However, at the light of the recent developments, mainly being the work by Costantini *et al.* [17], and at the light of the results got in this work, many useful properties of the stochastic processes and the pricing problems are provided. Such results have direct implication both from a theoretical point of view and from a practical perspective. In particular we have seen that, under quite general assumptions on the coefficients, and whenever a Lyapunov type condition is verified, then such processes do not reach the boundary of the domain or, in the case of unbounded domains the processes are not allowed to blow up in a finite time almost surely. Furthermore some estimates on the dependence on the initial data have been provided. In particular, even if the processes are driven by coefficients that are only locally Lipschitz continuous, then it is possible to prove a kind of continuity with respect to the initial data.
Furthermore, dealing with the problem of pricing derivatives, following the results provided in [17], we have seen that the valuation equation admits one and only one solution under general conditions, and a representation formula for $u(x,t)$ is also provided. However we have proven that such a viscosity solution, that is granted to be only continuous for the general case, can exhibit additional regularity in the subsets where the final payoff is regular enough. In particular, the solution $u(x,t)$ has been shown to be Hölder continuous in each subset where the final payoff is Hölder continuous. At the end, in all compact domains where the diffusion matrix is positive defined, the viscosity solution $u(x,t)$ is proven to be even twice differentiable with respect to x and once with respect to t.

In this chapter we try to apply the results got so far. In particular we will focus on the

model proposed by Ekströ̈m and Tysk in [29]. In their work, the authors present some interesting results dealing with stochastic volatility models, assuming general features for the drift term $\mu(x)$ and the diffusive one $\sigma(x)$. Then the well known Heston model is a particular case of their model. In particular we are interested in applying the results we have got in previous chapters, dealing with the pricing problem. In the study of the existence and uniqueness of the solution in the viscosity sense, it is possible to generalize their model, allowing the process X to have sudden jumps. Unfortunately, when we want to study additional regularity to the solution, our results can be applied to pure diffusive problems.
All the results we get in this chapter are substantially obtained assuming the coefficients μ and σ verify all the hypoteses assumed in [29]. On the other hand, our approach allows us to consider final payoffs ϕ that are more general with respect to ones considered by the authors.

The Chapter is organized as follows. In the first section an introduction on the model proposed in [29] is provided, pointing our attention on the form assumed for the stochastic process and on the assumptions that are made on the coefficients. Then two very interesting results on the existence and uniqueness of the solution provided by the authors are given. In the second section, instead, the explicit form for the Lyapunov function is given. In particular, we verify that Assumptions 4.3 is provided, and all Assumptions 4.1-4.4 and 5.1 are satisfied under the same hypoteses made in [29]. Hence, at the light of Theorems 5.3, 5.4 and Proposition 5.1 we can state some results of existence and uniqueness of the solutions in a classical sense. Furthermore, Theorems 5.3 and 5.4 allow us to state and prove existence and uniquenss of the viscosity solution in the case sudden jumps are included in the model.

Once all the assumptions of our theorems are verified, the existence and uniqueness of the solution and additional regularity is automatically achieved.

We can anticipate an interesting consideration on the Lyapunov function. In particular, whenever the evolution of the volatility term is assumed to evolve following a CIR model, then in a very natural way the well-known Feller's condition on the parameters κ, θ and σ_0 is needed in order to avoid the process V_t reaches the boundary.

6.1 Classical solution for stochastic volatility models

In last year Ekströ̈m and Tysk proposed in [29] a model that belongs to the class of the stochastic volatility models. In particular such a model is a Black and Scholes-type model, since the price of the asset is assumed to evolve following a GBM diffusion process, and the volatility is a stochastic process as well. As we have already remarked in the previous chapters, stochastic models in this generality may be not well defined. Furthermore, even if there exists a solution to the problem (2.9), existence, uniqueness and regularity of the solution to the problem (3.1) may be not granted. Then some simplifications on the coefficients are needed. In particular, the authors assume the functions $\beta(S_t, V_t) = \beta(V_t)$ and $\sigma(S_t, V_t) = \sigma(V_t)$ depend on V_t, as in the most of stochastic volatility models, but they allow to freely choose the form of such functions, provided that some assumptions on their regularity are verified. Such assumptions are obviously needed in order to guarantee the existence and regularity of the solution to the pricing problem (3.1). As it should be clear the model proposed in [29] generalizes the well-known Heston model.

In particular, in the model proposed by Ekström and Tysk in [29], the process (S_t, V_t) is assumed to evolve as

$$(ET)\begin{cases} dS_t = \sqrt{V_t}S_t dW_t^1 \\ dV_t = \beta(V_t)dt + \sigma(V_t)dW_t^2 \end{cases}$$

where W_t^1 and W_t^2 are two starndard brownian motions with constant correlation $\rho \in (-1,1)$. In terms of deterministic differential approach, the corresponding infinitesimal generator $\mathcal{A}_d$ is a pure diffusive operator of the form

$$\mathcal{A}_d g = \frac{1}{2} v s^2 \frac{\partial^2}{\partial s^2} g + \rho\sigma(v)\sqrt{v} s \frac{\partial^2}{\partial s \partial v} g + \frac{\sigma^2(v)}{2} \frac{\partial^2}{\partial v^2} g + \beta(v) \frac{\partial}{\partial v} g. \tag{6.1}$$

where $g \in C^{2,2}$.

In their work, some specific assumptions are made on the coefficients β and σ, that are required to be regular enough, in order to guarantee the existence and uniqueness of a classical solution to the pricing problem of the type (3.1), with infinitesimal operator given by $\mathcal{A}_d$ defined in (6.1). Furthermore, when a European type derivative with final payoff ϕ is considered, then some assumptions are obviously needed also for ϕ. In particular the following hypoteses are made.

Definition 6.1. *We say the functions β, σ, and ϕ verify the Assumption ET if the following hypoteses hold*

i) $\beta \in \mathcal{C}^1([0,\infty))$ *with α-Hölder continuous derivatives and* $\beta(0) \geq 0$

ii) The volatility $\sigma : [0,\infty) \to [0,\infty)$ verifies $\sigma(0) = 0$ and $\sigma(v) > 0$ for all $v > 0$ and the function $\sigma(v)^2$ is continuously differentiable on $[0,\infty)$.

iii) The functions β and σ have at most linear growth of rate, that is

$$|\beta(v)| + \sigma(v) \leq C(1+v)$$

iv) The payoff function ϕ is so that $s\phi'(s)$ and $s^2\phi''(s)$ are bounded.

The authors have shown in [29] that if β, σ and ϕ verify Assumption *ET* then, the price $u(x,t)$ of the European contigent claim with final payoff phi, defined as in (5.28) and with $c = f = 0$ is a classical solution to the pricing equation (3.1), provided that some boundary conditions are satisfied. Such a result is get in their Theorem 2.3 that can be formulated in the following way

Theorem 6.1 (Theorem 2.3, [29])**.** *Suppose Assumptions ET are satisfied. Consider the function $w(s,v,t)$ defined as*

$$w(s,v,t) = \mathbb{E}\left[\phi(X_T)\Big|\mathcal{F}_t\right]. \tag{6.2}$$

Then, the function w is a classical solution to the pricing equation

$$\begin{cases} \partial_t w(s,v,t) + \mathcal{A}_d w(s,v,t) = 0 & (s,v,t) \in (0,\infty)^2 \times [0,T) \\ w(0,v,t) = \phi(0) & (v,t) \in [0,\infty) \times [0,T] \\ \partial_t w(s,0,t) + \beta(0)\partial_v w(s,0,t) = 0 & (s,t) \in (0,\infty) \times [0,T] \\ w(s,v,T) = \phi(s) & (s,v) \in (0,\infty)^2. \end{cases}$$

Such a result is interesting since many cases of interest are included where the evolution of the type *(ET)* is considered. Unfortunately, the assumptions made on the final payoff seem to be quite restrictive, as stated also by the authors. In their work, the autors conjecture that the operator $\mathcal{A}$ may have sufficient regularizing properties if $\beta(0) > 0$. At the light of our results, it seems that such an additional assumption is necessary and almost sufficient in order to guarantee that the operator $\mathcal{A}_d$ has sufficient regularizing properties.

A very interesting result got by Ekström and Tysk deals with the uniqueness of the classical solution whenever the hypoteses in their Theorem 2.3 are verified.
In particular, the author prove that, even if Assumptions *(ET)* are satisfied, the uniqueness of the classical solution may be lost, and is granted just is the final payoff has at most strictly sublinear growth rate. If it is not the case, then some additional assumptions are needed. In particular, these assumptions deal with the correlation between the asset price and the volatility, or the growth rate of the volatility process. We see in details such assumptions. If such hypoteses are sutisfied, then the uniqueness of the classical solution can be achieved also for linear payoffs.

Such results are well formalized in the following theorems.

Theorem 6.2 (Theorem 6.1, [29])**.** *There is at most one classical solution to the pricing equation, which is of strictly sublinear growth in x and polynomial in y.*

Such a result is strongened by Andersen in [7] in which a specific example is shown. However, Ekström and Tysk have shown that, if the correlation $\rho \le 0$ or the growth rate of the volatility is less than half, then we can allow the payoff to be linear in x and polynomial in y in order achieved the uniqueness of the classical solution to the pricing equation. Such a result is well defined in their

Theorem 6.3 (Theorem 6.2, [29])**.** *Assume that $\rho \le 0$ or that $\sigma(v) \le C(1+v^\gamma)$ with $\gamma \le 1/2$ for all v and some constant C. Then there is at most one classical solution to the pricing equation in the class of functions that are at most linear in x and polynomial in y.*

From an intuitive point of view, this result can be easily explained. In the case of volatility models, the volatility of the process S_t is a stochastic process. If the volatility and the prices are positively correlated, then more the prices rise, more their volatility blows up. Then it is possible that the expected value of the process S_t could be not defined. If this condition occurs, then the expectation of a payoff having a linear growth rate with respect to S_t, or even higher, could not be defined. Hence a kind of boundness to the growth rate of the volatility is required, or the prices have to be negatively correlated with the level of the volatility.

For a detailed discussion of the results, and a useful background in the case of the model proposed by Ekström and Tysk, we refer directly to their work [29] and the reference therein.

6.2 Solutions for jump-stochastic volatility model

In this section we want to understand how our results can be applied to the model proposed by Ekström and Tysk and which results we can get if we consider less restrictive assumptions on the coefficients and the final payoff, or including sudden jumps both to the process representing the prices of the asset and the volatility. It is obvious that when sudden jumps are included Assumption 4.2 have to be verfied and the diffusion matrix $\sigma^2(v)$ is required to be twice differentiable and not only be locally Lipschitz continuous. We observe since now that, in the case of jump-stochastic volatility model, the results we have presented in this work allow us the deal with the existence and uniqueness of viscosity solutions, but no additional regularity is granted.

As it will be clear, even in a jump-diffusion stochastic volatility framework, whenever hypotheses *(ET)* on the coefficients and in Theorems 6.1 and 6.2 in [29] are fulfilled, under suitable assumptions on the final payoff ϕ it is possible to find a Lyapunov function V that verifies Assumptions 4.3 and 4.4.

Remark 6.1. *We observe that Proposition 8.24 in [9] implies that, whenever the hypoteses provided by Ekström and Tysk are satisfied then Assumption 4.1 is satisfied as well.*

Then, if we are able to prove that there exists V such that Assumptions 4.3 and 4.4 are satisfied, then we can apply the results of Theorem 5.3 and Theorem 5.4 in order to state that the pricing problem (3.1) admits one and only one viscosity solution.

On the other hand, in the case we consider just pure diffusive problems, we have shown in the previous chapters that under the same assumptions on the coefficients of the stochastic process X_t, some important results on the regularity of X with respect to initial data is granted. Furthermore, such a regularity is inhered by the solution of the pricing problem (3.1). In particular, in the case the running cost f and the final payoff ϕ have suitable properties of regularity that are verified in the most contracts traded in the real markets, the unique solution u is not only a solution in the sense of viscosity theory, but gains more regularity than the mere continuity.

At the end, we observe that the diffusion matrix $a = \sigma^2$ in [29] is positive defined in each compact subset of D. Hence, as a direct result of Theorem 5.7 and Proposition 5.1, we get that there exists a classical solution to (3.1) that coincides with the viscosity solution. On the other hand, if Assumptions 4.1-4.4 are satisfied, as in this case, then the process does not reach the boundary almost surely.
Hence, applying the results of Theorem 5.7 we can find, in an independent way, the same results provided in Theorems 6.1 and 6.2 in [29], even in the case of more general hypoteses on the final payoff ϕ.

Then, the presence of the jumps allows us to deal with just viscosity solutions, but, turning back to the original model, it is possible to achieve a classical solution to the pricing problem (3.1) considering assumptions on the final payoff that are weakened with respect to the ones proposed in [29].

We see that in order to guarantee the existence of such a Lyapunov function, an additional condition taking into account the behaviour of σ^2 and β when v approaches to zero is required. Hence, our aim in this section is to show the existence of a *family* of Lyapunov functions taking for granted the assumptions provided in [29]. We speak about a *family* of Lyapunov function since the specific form of such functions depends also on the particular final payoff ϕ.

First of all we consider a generalization of the model proposed in [29], allowing the state process $X_t = (S_t, V_t)$ to have sudden jumps. In particular, when we deal with the pricing problem (3.1) the additional nonlocal term has to be added to the pure differential operator.

Assumption 6.1. *Suppose that the measure $m(s,v,dz,dw)$ satisfies the following condition*

$$\int_{\mathbb{R}^2_+} h(z,w)m(s,v,dz,dw) \leq C\left(1+h(s,v)\right) \tag{6.3}$$

where $h(s,v) = P_{n,q}(s,v) - \ln(s) - \ln(v)$, and $P_{n,q}(s,v)$ has at most polynomial growth rate at infinity, of degree n with respect to s and q with respect to v.

The new integro-differential operator is then

$$\begin{aligned} \mathcal{A}g &= \frac{1}{2}vs^2\frac{\partial^2}{\partial s^2}g + \rho\sigma(v)\sqrt{v}s\frac{\partial^2}{\partial s\partial v}g + \frac{\sigma^2(v)}{2}\frac{\partial^2}{\partial v^2}g + \beta(v)\frac{\partial}{\partial v}g \\ &+ \int_D \left[g(z,w) - g(s,v)\right] m(s,v,dz,dw) \end{aligned} \tag{6.4}$$

where g is twice differentiable with respect to its arguments.

In what follows, due to the results in Theorems 6.1 and 6.2 in [29], we restrict ourselves to consider payoffs ϕ that are at most sublinear in s and polynomial in v. We state now our main result in this section

Proposition 6.1. *Let $\mathcal{A}$ be the integro-differential operator defined as in (6.4). We make the following assumptions, that are very closed to ones proposed in [29]*

i) The functions β and σ are locally Lipschitz continuous and have sublinear growth rate.

ii) The functions β and σ satisfy $\beta(0) \geq 0$, $\sigma(0) = 0$, and $\sigma(v) > 0$ for $v > 0$.

If jumps are allowed, that is $m(s,v,dz,dw) \neq \delta(s,v)dzdw$, then the diffusion matrix $a(v) = \sigma(v)^2$ is twice differentiable, and Assupntion 6.1 is satisfied for some $n \geq 1+\delta$ with $\delta > 0$ and $q \geq 3$.

iii) There exists a $c_0 < \infty$ such that the condition $\left(\frac{\sigma^2(v)}{2v^2} - \frac{\beta(v)}{v}\right) \leq c_0$ is hold true.

iv) The payoff ϕ is continuous and has at most striclty sublinear growth rate $r < 1$ in s and polynomial in v with degree p such that $q > 1 - r + \max(2,p)$.

As an alternative of [iv)] we can consider the following hypotesis

iv′) The payoff ϕ is continuous and has at most linear growth rate in s and polynomial in v with degree p, and the correlation ρ in the operator (6.4) is not positive, or the coefficient σ verifies $\sigma(v) \leq C(1+v^\gamma)$ with $\gamma \leq 1/2$.

Then there exists a Lyapunov function $V_\phi(s,v)$ dependent on the payoff ϕ such that all Assumptions 4.1-4.4 are verified.

Corollary 6.1. *There exists a unique viscosity solution u to the pricing problem (3.1), where the infinitesimal generator $\mathcal{A}$ is defined in (6.1).*

Corollary 6.2. *If the measure $m(s, v, dz, dw) = \delta(s, v)dzdw$ then there exists a unique classical solution v to the pricing equation (CET)*

$$(CET)\begin{cases} \partial_t v + \mathcal{A}_d v - cv = 0 & (x,t) \in S \times [0,T) \\ v(x,T) = u(x,T) & x \in S \\ v(x,t) = u(x,t) & (x,t) \in \partial S \times [0,T] \end{cases}$$

that coincides with the viscosity solution u of the pricing problem (3.1).

We can observe that Assumptions (ET) *i)* and *ii)* imply the Assumptions *i)* and *ii)* in Proposition 6.1. Furthermore the condition on the final payoff ϕ given in (iv) and in (iv') is obviously less restrictive with respect to the one proposed in the Assumption (ET).

In addition to the hypoteses *(ET)* we are forced to make the additional assumption *iii)* in order to control the behaviour of the volatility near the boundary.

Remark 6.2. *Whenever $\sigma^2(y)$ verifies Assumptions (ET), our the assumption (iii) in Proposition 6.1 is met if the following condition is satisfied*

$$\lim_{y \to 0^+} \frac{\left(\sigma^2\right)'(y)}{y} \leq \beta(0). \tag{6.5}$$

Such a remark suggests that in the case the condition $\beta(0) > 0$ is satisfied, then the operator $\mathcal{A}$ may have enough regularizing effect.

On the other hand, in the case of Corollary 6.2, the hypoteses on the final payoff ϕ are less restrictive with respect to the ones proposed by (ET). Furthermore the additional requirement on the behaviour of the solution near the boundary is not really needed.

Remark 6.3. *Assumption (iii) in Proposition 6.1 can be weakened of course, considering the following alternative $\left(\frac{\sigma^2(v)}{2v^2} - \frac{\beta(v)}{v}\right) \leq c_0(1 + V_\phi(s, v))$.*

It is interesting to observe that, in the case the volatility process follows a CIR model, then the Assumption (iii) becomes

$$\left(\frac{\left(\sigma_0\sqrt{v}\right)^2}{2v^2} - \frac{\kappa\left(\theta - v\right)}{v}\right) = \left(\frac{\sigma_0^2}{v} - \frac{\kappa\theta}{v} + \kappa\right) \leq c_0$$

for each $v \in [0, \infty)$, that implies the well known Feller's condition

$$\sigma_0^2 \leq 2\kappa\theta.$$

Such a result can be infered also from the relation (6.5).

As it has been already remarked, the uniqueness of the classical solution is lost for general payoff in the stochastic volatility models, and some particular hypoteses on the process are also needed. Such hypoteses are reflected in Assumptions *iv)* and *iv*$'$) of Proposition 6.1. We see that, in our approach, such assumptions are needed in order to ensure the existence of the Lyapunov function, and then the existence of the viscosity solution, so that the theory developed in the previous chapters remains still valid.

Proof of Proposition 6.1 We notice since now that Assumptions 4.1-4.2 are verified. Now we want to show that Assumptions 4.3 and 4.4 are satisfied as well. As we have already stated in Section 4.2 the Lyapunov function $V_\phi(x, y)$ determines the growth rate allowed for the final payoff ϕ. At reverse, this means that given a payoff ϕ it is not always possible to find a Lyapunov function V that verifies all Assumptions 4.1-4.4, even if the integro-differential operator is good enough.
At the light of Theorems 6.1 and 6.2 in [29] this remark is easily understandable. We state that the functions $V_{1^-}(s, v)$ and $V_1(s, v)$ defined as

$$V_{1^-}(s, v) = v^q + s + v - \ln(s) - \ln(v) \tag{6.6}$$

$$V_1(s, v) = V_{1^-}(s, v) + s^{1+\delta} v + s^{1+\delta} + sv^q \tag{6.7}$$

with $q = 1 - r + \max(2, p) + \epsilon$ and $\epsilon > 0$, $\delta > 0$ are two good Lyapunov functions for the integro-differential operator $\mathcal{A}$ that verify Assumptions 4.3 and 4.4, whenever we consider the payoffs $\phi_{1^-}(s)$ and $\phi_1(s)$ that verify respectivelly the assumptions *iv)* and *iv*$'$) in Proprosition 6.1.

It is clear, by the definition of $V_{1^*}(s, v)$ and the properties of the measure m that, for $(s, v) \to \partial D$ the proposed Lyapunov functions $V_{1^*}(s, v)$ verify

$$\lim_{(s,v)\to\partial D} V_{1^*}(s, v) = +\infty, \tag{6.8}$$

$$\lim_{|(s,v)|\to+\infty} V_{1^*}(s, v) = +\infty, \tag{6.9}$$

$$\int_D V_{1^*}(z, w) m(s, v, dz, dw) < +\infty \tag{6.10}$$

where the last inequality follows from the assumptions on the measure m and Hölder inequality. Hence if we are able to verify the relation

$$\mathcal{A}V(s, v) \leq C\,(1 + V(s, v))\,,$$

we get that Assumption 4.3 is verified. At the end it is left to prove the equation (4.17) in

Assumption 4.4.

We first consider the function $V_{1^-}(s,v)$. It is easy to see that, applying the operator $\mathcal{A}$ to the function $V_{1^-}(s,v)$, by the hypotesis *iii)* in Proposition 6.1, we have

$$\begin{aligned}\mathcal{A}V_{1^-}(s,v) &= \frac{q(q-1)}{2}\sigma^2(v)v^{q-2}+\frac{1}{2}v+\left(\frac{\sigma^2(v)}{2v^2}-\frac{\beta(v)}{v}\right)+\beta(v)\left(1+qv^{q-1}\right)\\ &+ \int_D V_{1^-}(z,w)m(s,v,dz,dw)-m(s,v,D)V_{1^-}(s,v)\\ &\leq C_1\left(1+v^q+v+s+V_{1^-}(s,v)\right)+\left(\frac{\sigma^2(v)}{2v^2}-\frac{\beta(v)}{v}\right)\\ &\leq C_1\left(1+V_{1^-}(s,v)\right). \end{aligned} \tag{6.11}$$

where the inequalities come from the sublinear growth of β and σ, the relation (6.10), and the hypotesis *iii)* of Proposition 6.1.

Now we have to show equation (4.17) in Assumtpion 4.4, that is

$$|\phi_{1^-}(s,v)|\varphi(|\phi_{1^-}(s,v)|)\leq C(1+V_{1^-}(s,v)) \tag{6.12}$$

where the function φ is the function in Assumption 4.4.

Since the payoff $\phi_{1^-}(s,v)$ has at most strict sublinear growth $r<1$ with respect to s, then the function $\varphi(z)=z^\alpha$ for $\alpha=1-r>0$ verify Assumption 4.4. Indeed, in such a case, we have

$$\begin{aligned}|\phi_{1^-}(s,v)|\varphi(|\phi_{1^-}(s,v)|) &= |\phi_{1^-}(s,v)|^{1+\alpha}\\ &\leq C_{\phi^-}\left(1+|s|+|v|^{p+1-r}\right)\\ &\leq C\left(1+V_{1^-}(s,v)\right),\end{aligned}$$

where the last inequality comes from the definition of q. This means that V_{1^-} is a Lyapunov function in the sense of [17]. Then all Assumptions 4.1-4.4 are verified, and by applying the result of Theorems 5.3 and 5.4, we get that there exists a unique viscosity solution $u(x,t)$ to the pricing problem (3.1) when a generalization of the Ekstrom̈ and Tysk operator is considered and jump diffusive processes are allowed. Furhtermore, as a consequence of Theorem 4.3, the stochastic process X associated to the infinitesimal generator $\mathcal{A}$ does not reach the boundary of the domain $[0,\infty)^2$ in a finite time.

Remark 6.4. *The considered pricing problem does not need the boundary condition.*

Now we consider the payoff $\phi_1(s)$ that verify the assumption *iv′)* and the Lyapunov function

$V_1(s,v)$. In order to end the prove of Proposition 6.1 we have to show the relation

$$\mathcal{A}V_1(s,v) \le C\left(1+V_1(s,v)\right),$$

and the equation (4.17) in Assumption 4.4, as in the previous case. In such a linear case we can perform exaclty the same calculation as in the relation (6.11) provided that the assumption *iv′)* is hold true. Our results are in line with the ones got by Ekströ̈m and Tysk in Theorem 6.2 [29] as in the sublinear case, since we can write

$$\begin{aligned}
\mathcal{A}V_1(s,v) &= \frac{q(q-1)}{2}\sigma^2(v)v^{q-2}\left(1+s\right)+\frac{1}{2}v+\left(\frac{\sigma^2(v)}{2v^2}-\frac{\beta(v)}{v}\right)\\
&+ \beta(v)\left(1+qv^{q-1}\left(1+s\right)\right)+\delta(1+\delta)\frac{s^{1+\delta}v}{2}+\rho\sigma(v)\sqrt{v}s^{1+\delta}\\
&+ \int_D V_1(z,w)m(s,v,dz,dw)-m(s,v,D)V_1(s,v)\\
&\le C_1\left(1+v^q+v+s^{1+\delta}v+s+V_1(s,v)\right)+\left(\frac{\sigma^2(v)}{2v^2}-\frac{\beta(v)}{v}\right)\\
&\le C_1\left(1+V_1(s,v)\right).
\end{aligned} \tag{6.13}$$

At this point the last part of Proposition 6.1 is trivially achieved since we can consider the function $\varphi(z)=z^\delta$. In such a case, in fact, since the payoff $\phi_1(s,v)$ is sublinear in s, then we have

$$\begin{aligned}
|\phi_1(s,v)|\varphi(|\phi(s,v)|) &\le C_\phi\left(1+|s|+|v|^p\right)\left(|s|^\delta+|v|^\delta\right)\\
&\le C\left(1+V_1(s,v)\right)
\end{aligned} \tag{6.14}$$

where the last inequlity comes again from the definition of q.■

Hence, Remark 6.1 can be proven as a direct application of Theorems 5.3 and 5.4 in the previous Chapter.

Proof of Corollary 6.2 This is a direct application of Proposition 6.1 and Proposition 5.1. Indeed in force of Proposition 6.1 the existence of a unique viscosity solution is granted. Then the thesis comes by observing that the diffusion matrix $a(s,v)$ verifies the hypoteses in Theorem 5.7 and Proposition 5.1. This means that there exists a unique classical solution to the pricing problem (CDP) that coincindes with the viscosity solution (5.28).■

Remark 6.5. *We remark that the results of Proposition 6.1 are in line with the ones obtained by Ekström and Tysk in Theorems 6.1 and 6.2 in [29].*

Remark 6.6. *We observe that even in the case the diffusive matrix $\sigma(v)$ is assumed to be not uniformly positive defined, then such a result of existence, uniqueness and regularity of* *the solutions could be applied. In this case a global unique viscosity solution is granted by the results provided in [17]. Furthermore, some additional regularity could be found by applying the results we have got in this work, whenever suitable assumptions are satisfied.*

6.3 Conclusions

In this chapter we have introduced a very general model belonging to the class of stochastic volatility models proposed by Ekstr̈om and Tysk in their work [29]. Their model can be seen as a generalization of the most known Heston model, that is estensively used in the financial markets. In their work, the authors are able to show under some assumptions on the coefficients μ and σ, and the final payoff ϕ, there exists a classical solution to the pricing problem (3.1). Furthermore, such a solution is unique whenever the final payoff satisfies additional properties, namely it is required to not blow up fastly at infinity and regular enough.

On the other hand, the results we have got in the previous chapters represent a very useful tool in order to investigate the existence, uniqueness and regularity of the solution to the pricing problem. In particular we have seen, in an independent way that, whenever their assumptions on the coefficients and the final payoff are satisfied, we get the same results in terms of classical solutions. Furthermore we are able to consider final payoffs that satisfy less restrictive assumptions with respect to the ones allowed in [29]. Indeed it is possible to consider final payoffs that are required to be just continuous and with sublinear growth rate, and no boundary conditions are needed.
Furthermore, we are able to deal with processes that exhibit sudden jumps. In such a case we are able to state and prove that, whenever the measure of the jumps satisfies suitable hypoteses, the pricing problem driven by the considered stochastic process admits a unique solution in the framework the viscosity solution theory.

Chapter 7

Conclusions

We have seen that a powerful approach for the description of the markets is represented by the commonly called *market theory*. In particular, we have seen how it is possible to define in a mathematical framework the players acting in the market and the products that are traded. This approach is expecially useful in force of the remarkable results available from the stochastic theory, and can be succesfully applied to the most liberalized market such as financial, most of commodity markets and so on and so forth. Unfortunately, the solution of the Stochastic Differential Equations considered for some market model are often not available, and the existence and uniqueness of the fair price of a given contract is sometimes hidden to our knowledge.
On the other hand, it is well-known that it is possible to study the existence and uniqueness of the fair price by solving the corresponding Partial Integro-Differential Equation. Unfortunately, the literature available dealing with Partial Integro-Differential problems may be not able to deal with many problems, that are of interest in modern finance. This is expecially true when the nonlocal term is present, and the operator is not purely differential. In last years several works have been made in order to overcome some issues. All these works deal with some specific assumptions on the form of the differential operator or on the coefficients of the stochastic process. All these approaches allow to deal with some specific problem in finance.

We have seen in Chapter 3 the assumptions that are actually required in order to ensure the existence and uniqueness of the solution to the pricing problem (3.1). In some of them additional results of the regularity of the solution is also provided. However, when the coefficients of the SDE (2.9), or the final payoff does not fit such assumptions, the existence, uniqueness and regularity of the solution may be not guranteed. In particular this is the case when the diffusion matrix $a(x)$ is singular in some subsets of the domain or the drift μ and the matrix σ are not Lipschitz continuous up to the boundary of the domain D, or they are fast growing near the boundary or at infinity. Furthemore, some issues can take place if

the considered problem (3.1) has not conditions on ∂D, or they are not specified. This is the case, as an instance, when numerical procedures are put in place in order to solve the problem, and the considered domain is truncated. In these cases it may not be clear which boundary conditions have be considered.
We observe that all the previous cases are met in several problems of interest among practitioners in finance, in particular when some stochastic volatility models are considered and for the case of Asian options. However, for these cases, results of existence and uniqueness of the viscosity solution is provided in [17].
We observe that in last years, some works have been made that conjecture that in most case of interest, boundary conditions are not really needed, and they are redundant from a strict mathematical point of view (see e.g. [39] and [29]). Hence it is common to speak about the *behaviour near the boundary* instead of *boundary conditions*.

On the other hand, we have seen in Chapter 4 a recent result showed in [17]. In particular it ensures that, under very general conditions that include the cases we have just mentioned, the boundary of the domain D is prohibited to the process X. In particular, such a feature is met when the coefficients μ, σ and the measure m satisfy Assumptions 4.1-4.4. Furthermore, in the same work the results of the well-posedness of the martingale problem $(\mathcal{A}, P_0)$ for any initial distribution P_0 is given. The existence of a unique strong solution to the stochastic differential equaiton with jumps (2.9) is often taken for granted. On the other hand, when assumptions on the coefficients are weakened it is not clear even if there exists a unique solution in a weak sense. Hence the result provided in [17] are useful for our purposes. In particular, in Chapter 4 we have proven some estimates on the dependence of the solution X_t^x with respect to the initial data for the purely differential case. In particular, as for the Ordinary Differential Equations, it is possible to prove that whenever the coefficients are locally Lipschitz continuous, the solution X is Lipschitz continuous up to a suitable stopping time τ. This stopping time represents the instant at which the process X exits a fixed domain K contained in D.

A result of uniform continuity with respect to x in a suitable sense is also provided for any time t. In particular, the existence of the Lyapunov-type condition plays a crucial role.

At the end, we have proven that the process X is Hölder continuous, in a proper sense, with respect to the time t, for the stopping time τ that is the exit time from a given domain. Furthermore, in Section 4.4 we have shown that, the Lipschitz-type dependence with respect to x and Hölder-type dependence with respect to t is ensured for each time $T > 0$ if a suitable weight function is considered. The weight function is directly connected to the coefficients μ and σ and it depends on the rate at which such coefficients loose their regularity, that is when the stochastic process X approaches the boundary of the domain, where the Lipschitz continuity is definetely lost. Also in this case, if the coefficients μ and σ are globally Lipschitz continuous, then such a result coincides with the standard ones for Lipschitz continuous coefficients.

On the other hand, the regularity of the solution to the pricing problem (3.1) has been also studied under weak assumtpions on the coefficients and final payoff. This topic is studied in Chapter 5. We have discussed in Chapter 3 the reasons why the knowledge of the existence, uniqueness and regularity of the solution for the pricing problem (3.1) is of crucial importance. In particular, even if the existence and uniquenss of such a solution is theorically known, it is generally not known explicitely and some numerical methods have to be put in place. Furthermore, when the solution of the problem is smooth enough, then the numerical procedures fastly increase their rate of convergence towards the analytical solution.

Chapter 5 introduces the problem of existence and uniqueness of the solution for a very simple case of singular valuation equation. Then, the general pricing problem is considered. In particular, the results provided in [17] are proposed. Such results ensure that, under Assumptions 4.1-4.4 the existence of a unique viscosity solution to the general problem (3.1) is proven. Unfortunately the viscosity solution theory ensures just that the solutions are continuous but no further regularity is granted a priori. However, such results are of crucial importance for our work. In particular, applying the results we have proven in Chapter 4, it is possible to show that the viscosity solution $u(x,t)$ is not only a mere continuous function but is also a classical solution in each compact subset K where the diffusion matrix is uniformly positive defined, and the running cost and the interest rate are Hölder continuous, the viscosity solution is twice differentiable with respect to x, and once with respect to t.

Chapter 6 is devoted to give an application of the results got in the previous chapters. In particular a focus on the model proposed by Ekström and Tysk in [29] is given, and the main improvements with respect to the previous models are presented. Then, we have proven that, whenever some suitable final payoffs ϕ are considered, Assumptions 4.1-4.4 are satisfied by the model propesed in [29]. In particular an explicit expression for the Lyapunov function $V(x)$ is given. As a direct consequence, all the results we have got in previous chapters can be applied, and the existence of a classical solution to the problem (3.1) is given. In particular we observe that in our framework, the boundary of the domain D is forbitten to the process X, hence the boundary conditions for the problem (3.1) are not really needed. We observe furthermore that the form of the Lyapunov function $V(x)$ is still valid if we consider a generalization of the model, allowing the state process X to exhibit sudden jumps. In this case, applying the results provided in [17], we are able to prove the existence of a unique viscosity solution to the considered problem when the final payoff does not blow up fastly at infinity.

We observe that further work could be made in order to better understand under which conditions the integro-differential operator has enough regularizing properties. In particular our approach allows to consider just localized problems. Indeed, it is possible to argue that the existence of a unique viscosity solution $u(x,t)$ is a global property of the considered problem. On the other hand, the regularity of the solution is a local property of $u(x,t)$. In particular, the function $u(x,t)$ may exhibit some additional regularity in some subdomains where the operator has enough regularizing properties, that are not sufficient to guarantee the existence of a unique global regular solution $u(x,t)$. On the other hand, the presence of the nonlocal term may affect this approach, since the solution $u(x,t)$ is then connected to the whole domain D.

We consider furthermore that, the existence of the Lyapunov function could be used in order to get some useful informations on the behaviour of the stochastic process near the boundary. This additional knowledge could be enough to ensure that the stochastic process X is regular with respect to initial data. In particular, if the probability that the process X reaches the boundary of the domain vanishes fastly enough, it could be sufficient to ensure that the terms coming from the regions that may affect the regularity of the process X are neagleactable.
We have already said that, when some numerical methods are put in place in order to get a numerical solution, a truncation of the domain have to be done. Often it is not clear which "artificial" boundary conditions are suitable, then it could be interesting to understand if it is possible to get some estimates on such boundary conditions starting from our knowledge on the Lyapunov function. Hence, the Lyapunov function could give some indications on the domain that can be truncated without affecting the precision of the numerical solution.

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